CHARMS Summer School in Versailles

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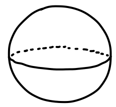
A motivating example



$\rightsquigarrow D^b\mathrm{Coh}(\mathbb{P}^1) \simeq D^b\left(\cdot \rightrightarrows \cdot\right)$ gentle!

$\mathcal{O}(-1)$, $\mathcal{O}$

$\mathcal{O}(-1)\oplus\mathcal{O}$: tilting

$\mathbb{P}^1$

projective line

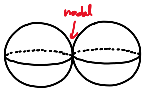
Goal : Generalize this in 2 ways:

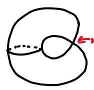
① gluing
② weighted projective lines

① gluing copies of $\mathbb{P}^1$ ref) [Burban-Drozd '11] Tilting on non-commutative rational projective curves

ex)

nodal





nodal

$\{xy=0\} \subset \mathbb{P}^2$  $\{y^2=x^3+x^2\} \subset \mathbb{P}^2$ 

singular Weierstraß cubic curve

In general, let

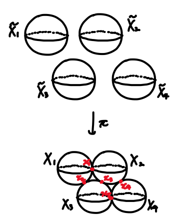
X: gluing of n-copies of $\mathbb{P}^1$ at finite nodal points $x_1,\dots,x_t$

i.e., $\exists\ \tilde{X} = \coprod_{i=1}^{n} \tilde{X}_i$ ← copies of $\mathbb{P}^1$

$\pi \downarrow$ normalization

$X = \bigcup_{i=1}^{n} X_i$ ← irreducible components of X

ex)



$\mathcal{O} := \mathcal{O}_X$ structure sheaf of X

$\cap$

$\tilde{\mathcal{O}} := \pi_*(\mathcal{O}_{\tilde{X}}) = \underbrace{\pi_*(\mathcal{O}_{\tilde{X}_1})}_{\tilde{\mathcal{O}}_1} \oplus\cdots\oplus \underbrace{\pi_*(\mathcal{O}_{\tilde{X}_n})}_{\tilde{\mathcal{O}}_n}$

$\mathcal{I}$: ideal sheaf of $\mathrm{Sing}(X) = \{x_1,\dots,x_t\}$

Consider the sheaf

$\mathcal{A} := \mathcal{A}_X := \mathcal{E}nd_X(\mathcal{I}\oplus\mathcal{O})$ Auslander sheaf on X

$$\simeq \begin{array}{c|cc} & \mathcal{I} & \mathcal{O} \\ \hline \mathcal{I} & \tilde{\mathcal{O}} & \mathcal{I} \\ \mathcal{O} & \tilde{\mathcal{O}} & \mathcal{O} \end{array}$$

We have 3 ringed spaces on X:

$X = (X, \mathcal{O})$ original curve — $D^b\mathrm{Coh}(X)$

$\tilde{X} = (X, \tilde{\mathcal{O}})$ hereditary cover of X ← $\mathrm{gl.dim}(\mathrm{Coh}(\tilde{X})) = 1$ $\rightsquigarrow$ $D^b\mathrm{Coh}(\tilde{X})$ relation? tilting?

$\mathbb{A} = (X, \mathcal{A})$ Auslander curve of X ← $\mathrm{gl.dim}(\mathrm{Coh}(\mathbb{A})) = 2$ — $D^b\mathrm{Coh}(\mathbb{A})$

Relations

(1) $F := \begin{pmatrix} \mathcal{I} \\ \mathcal{O} \end{pmatrix} \in \mathrm{Coh}(\mathbb{A})$ (sheaf of $\mathcal{A}$-modules)

$X \leftrightarrow \mathbb{A}$

$= \mathcal{I}\oplus\mathcal{O} \in \mathrm{Coh}(X)$ (sheaf of $\mathcal{O}$-modules)

$$\Rightarrow \begin{tikzcd} \mathrm{Coh}(\mathbb{A}) \arrow[r, "\mathbb{G} = \mathcal{H}om_{\mathcal{A}}(F,-)"] & \mathrm{Coh}(X) \arrow[l, bend right, "\mathbb{F} = F\otimes_{\mathcal{O}} -"'] \arrow[l, bend left, "\mathbb{H} = \mathcal{H}om_{\mathcal{O}}(F^{\vee},-)"] \end{tikzcd}$$

← fully faithful

& $(\mathbb{F}, \mathbb{G}, \mathbb{H})$: adjoint triple, $\mathbb{G}\circ\mathbb{F} \cong \mathrm{id}_{\mathrm{Coh}(X)}$

$$\Rightarrow \begin{tikzcd} \mathrm{Perf}(X) \arrow[r, "L\mathbb{F}"] & D^b\mathrm{Coh}(\mathbb{A}) \arrow[r, "D\mathbb{G}"] & D^b\mathrm{Coh}(X) \end{tikzcd}$$

← fully faithful ← some localization

categorical resolution of $D^b\mathrm{Coh}(X)$ i.e.

- $L\mathbb{F}$ is left adjoint to $D\mathbb{G}$ on $\mathrm{Perf}(X)$
  $\mathrm{Hom}_{D^b(\mathbb{A})}(L\mathbb{F}(A^\cdot), B^\cdot) \cong \mathrm{Hom}_{D^b(X)}(A^\cdot, D\mathbb{G}(B^\cdot))$, $\forall A^\cdot \in \mathrm{Perf}(X)$, $B^\cdot \in D^b\mathrm{Coh}($
- $D\mathbb{G}\circ L\mathbb{F} \cong \mathrm{id}_{\mathrm{Perf}(X)}$

$\wr$ $\wr$ HMS $\wr$

cf) $\mathrm{Fuk}(\Sigma) \longrightarrow \mathrm{WFuk}(\Sigma, \Lambda) \longrightarrow \mathrm{WFuk}(\Sigma)$ in A-side [Lekili & Polishchuk '17] Auslander orders over nodal stacky curves and partially wrapped Fukaya categories

(2) $P := \begin{pmatrix} \tilde{\mathcal{O}} \\ \tilde{\mathcal{O}} \end{pmatrix} \in \mathrm{Coh}(\mathbb{A})$

$\mathbb{A} \leftrightarrow \tilde{X}$

$= \tilde{\mathcal{O}} \oplus \tilde{\mathcal{O}} \in \mathrm{Coh}(\tilde{X})$

$\tilde{\mathbb{F}} = P \otimes_{\tilde{\mathcal{O}}} -$ ← fully faithful

$$\Rightarrow \quad \begin{tikzcd} \mathrm{Coh}(\mathbb{A}) \arrow[rr, "\tilde{\mathbb{G}} = \mathcal{H}om_{\mathbb{A}}(P, -)"] & & \mathrm{Coh}(\tilde{X}) \arrow[ll, bend right, "\tilde{\mathbb{F}} = P \otimes_{\tilde{\mathcal{O}}} -"'] \arrow[ll, bend left, "\tilde{\mathbb{H}} = \mathcal{H}om_{\tilde{\mathcal{O}}}(P^{\vee}, -)"] \end{tikzcd} \quad \& \quad (\tilde{\mathbb{F}}, \tilde{\mathbb{G}}, \tilde{\mathbb{H}}) : \text{adjoint triple}$$

$$\Rightarrow \quad \begin{tikzcd} \langle S_1, \dots, S_t \rangle \arrow[rr, "\mathbb{I}"] & & D^b\mathrm{Coh}(\mathbb{A}) \arrow[ll, bend right, "\mathbb{I}^*"'] \arrow[ll, bend left, "\mathbb{I}^!"] \arrow[rr, "D\tilde{\mathbb{G}}"] & & D^b\mathrm{Coh}(\tilde{X}) \arrow[ll, bend right, "L\tilde{\mathbb{F}}"'] \arrow[ll, bend left, "R\tilde{\mathbb{H}}"] \end{tikzcd} \quad \text{recollement diagram}$$

S_j : the torsion $\mathbb{A}$-module supported at the singular point x_j with $(S_j)_{x_j} \cong \mathbb{k}$

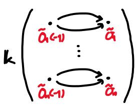
$\Rightarrow \quad D^b\mathrm{Coh}(\mathbb{A}) = \langle \mathrm{im}\,\mathbb{I}, \mathrm{im}\,L\tilde{\mathbb{F}} \rangle = \langle S_1, \dots, S_t, D^b\mathrm{Coh}(\tilde{X}) \rangle$ semi-orthogonal decomposition of $D^b\mathrm{Coh}(\mathbb{A})$

<u>Tiltings</u>

(1) $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_1 \oplus \cdots \oplus \tilde{\mathcal{O}}_n$ where $\tilde{\mathcal{O}}_i = \pi_*(\mathcal{O}_{\tilde{X}_i})$, $\tilde{X}_i$: $\mathbb{P}^1$

$\forall d \in \mathbb{Z}, \quad \tilde{\mathcal{O}}(d) := \tilde{\mathcal{O}}_1(d) \oplus \cdots \oplus \tilde{\mathcal{O}}_n(d)$ where $\tilde{\mathcal{O}}_i(d) = \pi_*(\mathcal{O}_{\tilde{X}_i}(d))$, $\tilde{X}_i$: $\mathbb{P}^1$

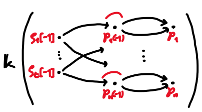
$\Rightarrow \tilde{\mathcal{O}}(-1) \oplus \tilde{\mathcal{O}}$: tilting on $D^b\mathrm{Coh}(\tilde{X})$

with $\mathrm{End}_{D^b(\tilde{X})}(\tilde{\mathcal{O}}(-1) \oplus \tilde{\mathcal{O}}) \cong \mathbb{k}$ 

(2) $S := S_1 \oplus \cdots \oplus S_t \in \mathrm{Coh}(\mathbb{A})$

$$P(d) := \tilde{\mathbb{F}}(\tilde{\mathcal{O}}(d)) = \begin{pmatrix} \tilde{\mathcal{O}}(d) \\ \tilde{\mathcal{O}}(d) \end{pmatrix} = \underbrace{\begin{pmatrix} \tilde{\mathcal{O}}_1(d) \\ \tilde{\mathcal{O}}_1(d) \end{pmatrix}}_{P_1(d)} \oplus \cdots \oplus \underbrace{\begin{pmatrix} \tilde{\mathcal{O}}_n(d) \\ \tilde{\mathcal{O}}_n(d) \end{pmatrix}}_{P_n(d)} \in \mathrm{Coh}(\mathbb{A})$$

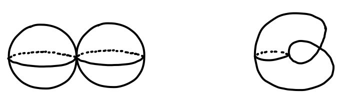
$\Rightarrow S[-1] \oplus P(-1) \oplus P$: tilting on $D^b\mathrm{Coh}(\mathbb{A})$

with $T_{\mathbb{A}} := \mathrm{End}_{D^b(\mathbb{A})}(S[-1] \oplus P(-1) \oplus P) \cong \mathbb{k}$ 

Keller's thm

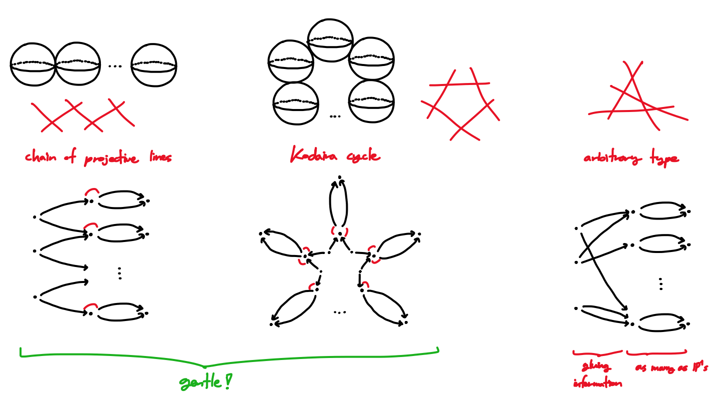
$\Rightarrow D^b\mathrm{Coh}(\mathbb{A}) \simeq D^b(\mathrm{mod}\text{-}T_{\mathbb{A}})$

ex)

X 

$\mathbb{A}$: Auslander curve of X

$T_{\mathbb{A}}$ 

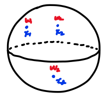


② Weighted projective lines

$x_1, \dots, x_\ell \in \mathbb{P}^1$ distinct points

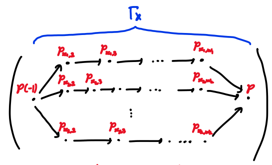
$w_1, \dots, w_\ell \in \mathbb{N}_{\geq 2}$ weight

$\mathbb{X} = \mathbb{P}^1(\vec{x}, \vec{p})$ weighted projective line



[Geigle & Lenzing '87]

· $D^b\mathrm{Coh}(\mathbb{X}) \simeq D^b$



w/ many relations

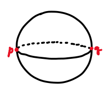
(Ringel's canonical algebra)

· $\mathbb{X}$ is hereditary $\left(gl.\dim(\mathrm{Coh}(\mathbb{X})) = 1 \right)$

$\Rightarrow D^b\mathrm{Coh}(\mathbb{X}) \simeq D^b\mathrm{Coh}(\tilde{\mathbb{X}}) \simeq D^b\mathrm{Coh}(A_{\mathbb{X}})$

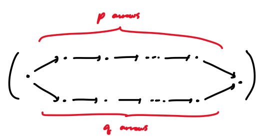
hereditary cover $\tilde{\mathbb{X}} = \mathbb{X}$ — Auslander curve of $\mathbb{X}$

· $\Gamma_{\mathbb{X}}$ gentle $\Leftrightarrow \ell = 2$



$\mathbb{X} = \mathbb{P}^1(p,q)$

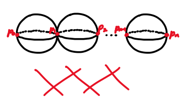
$\rightsquigarrow D^b\mathrm{Coh}(\mathbb{X}) \simeq D^b$



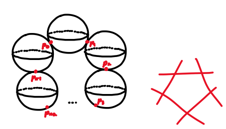
no relations

We can also glue these:

$\mathbb{X}$



nodal stacky chain

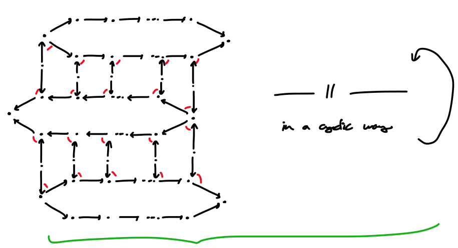


nodal stacky cycle

[Lekili & Polishchuk '17] Auslander orders over nodal stacky curves and partially wrapped Fukaya categories

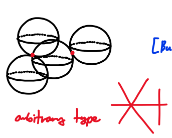
A : Auslander curve of $\mathbb{X}$

Γ_A



gentle? but not exhaust all of them

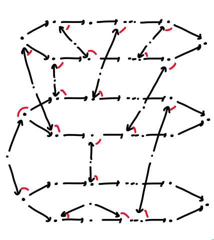
$\mathbb{X}$



[Burban-Drozd '19] Non-comm. nodal curves and derived tame algebras

A : Auslander curve of $\mathbb{X}$

Γ_A



gentle

These are the only tame non-comm. proj. curves of gentle type.

$\Rightarrow D^b\mathrm{Coh}(A) \simeq D^b(\mathrm{mod}\text{-}\Gamma_A)$