Intro to Gentle Alg.

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§1. Gentle algebras

- Assem and Skowroński (1987) introduced 'gentle algebra' to study 'iterated tilted algebra' of type A.

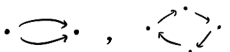
Def) A quiver pair (Q, I) is <u>gentle</u> if
(Q_0, Q_1, s, t)

- for any $v \in Q_0$, $\#s^{-1}(v), \#t^{-1}(v) \leq 2$,
- for any $\alpha \in Q_1$, there is at most one arrow β such that $t(\beta) = s(\alpha)$ and $\beta\alpha \notin I$.
- for any $\alpha \in Q_1$, " β' " $t(\beta') = s(\alpha)$ and $\beta'\alpha \in I$
- for any $\alpha \in Q_1$, " γ " $t(\alpha) = s(\gamma)$ and $\alpha\gamma \notin I$.
- for any $\alpha \in Q_1$, " γ' " $t(\alpha) = s(\gamma')$ and $\alpha\gamma' \in I$

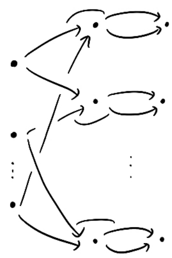
A k-algebra is <u>gentle</u> if it is Morita equivalent to $\Bbbk Q/I$ for some gentle quiver pair (Q, I).

Ex) $\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet \quad I = \emptyset$

$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet \quad I = \{\alpha\beta\}$

 ← ∞-dim'l

You will see more complicated examples in the next lecture by Kyungmin related with algebraic geometry :



Some properties :

Def) A <u>path</u> $p = \alpha_1 \cdots \alpha_n$ in (Q, I) is a path in Q such that $\alpha_i\alpha_{i+1} \notin I$ for any $i = 1, \dots, n-1$. (Equivalently, $p \neq 0$ in $\Bbbk Q/I$)

It is <u>maximal</u> if there is no path properly containing p.

prop) · Any arrow (or any path) is contained in a unique maximal path.

- $\{\text{paths}\} \cup \{\text{trivial paths}\}$ is a basis for $\Bbbk Q/I$.
- Any path in Q can be written as a product of paths $p_1 \cdots p_n$ in a unique way so that $p_i p_{i+1} = 0$ in $\Bbbk Q/I$

Ex) $\begin{tikzcd} \bullet \arrow[r, "\alpha"'] & \bullet \arrow[r, "\beta"'] & \bullet \arrow[r, "\gamma"'] & \bullet \arrow[lll, bend right, "\delta"'] \end{tikzcd} \quad I = \{\delta\alpha, \gamma\delta\}$

maximal paths : $\alpha\beta\gamma$, δ

$\beta\gamma\delta\alpha\beta \Rightarrow (\beta\gamma)(\delta)(\alpha\beta)$.

Thm) (Schröer - Zimmermann, 2003)

Let A be a finite dimensional gentle algebra. Then, any algebra derived equivalent to A is gentle.

In other words, the class of gentle algebras is closed under derived equivalence.

Appendix) Check SZ theorem for a simple example by hand.

Thm) (Schröer - Zimmermann, 2003)

Let A be a finite dimensional gentle algebra. Then, any algebra derived equivalent to A is gentle.

In other words, the class of gentle algebras is closed under derived equivalence.

- Derived equivalence and tilting object.

Def) An object (or a complex) $T \in \mathrm{perf}(A)$ is a tilting object if

- $\mathrm{Hom}_{D^b(A)}(T, T[n]) = 0$ for any $n \neq 0$,
- $\langle T \rangle = \mathrm{perf}(A)$.

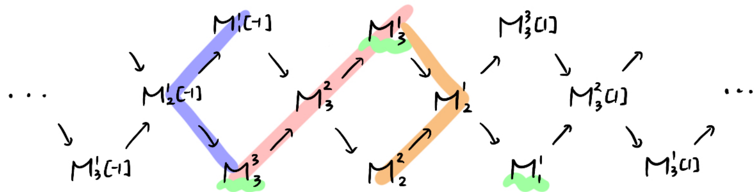
Thm) [Happel] For a tilting complex T of A, there is a derived equivalence $D^b(A) \xrightarrow{\sim} D^b(\mathrm{End}_{D^b(A)}(T))$.

Thm) [Rickard] If A and B are derived equivalent, then there is a tilting complex T of A such that $\mathrm{End}_{D^b(A)}(T) \cong B$.

Ex) Consider $A = k(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)$. We know the full list of indecomposable objects of $D^b(A)$ (up to shift):

Let us denote by M_j^i the representation $(M_j^i)_k = \begin{cases} K & i \le k \le j, \\ 0 & \text{o.w.} \end{cases}$

Then, the AR-quiver of A is,



So we have four types of tilting objects :

These are all gentle!?

So, an algebra is derived equivalent to A iff it is a gentle algebra one of

$K^{-,b}(\mathrm{Proj}(\Bbbk Q/I))$
$\cong$

§2. Indecomposables in $D^b(\Bbbk Q/I)$ for a gentle pair (Q, I).

- Bekkert and Merklen (2003) gave a classification of indecomposable objects of bounded derived category of gentle algebra.

Thm) There is one-to-one correspondence between $\mathrm{ind}\, D^b(\Bbbk Q/I)$ and $\{$generalized strings$\} \cup \{$generalized bands$\} \cup \{$non-perfect things$\}$ up to degree shift.

Rmk) Opper, Plamondon, and Schroll (2018 arXiv) found a geometric model for the classification.

- Introduce a 'formal inverse' $\bar{\alpha}$ for each $\alpha \in Q_1$ with $\begin{cases} s(\bar{\alpha}) := t(\alpha) \\ t(\bar{\alpha}) := s(\alpha) \end{cases}$, and let $\bar{\bar{\alpha}} = \alpha$. $\bar{Q}_1 := \{\bar{\alpha} : \alpha \in Q_1\}$.
- For a path $p = \alpha_1 \cdots \alpha_n$, let $\bar{p} = \bar{\alpha}_n \cdots \bar{\alpha}_1$. $\mathcal{P} := \{$paths in $(Q, I)\}$ and $\bar{\mathcal{P}} := \{$formal inverses of paths in $(Q, I)\}$.

⊙ Generalized strings

Def) A <u>generalized string</u> is a sequence of (formal inverses of) paths $w = w_1 \cdots w_n$ ($w_i \in \mathcal{P} \cup \bar{\mathcal{P}}$) such that

- if $w_i, w_{i+1} \in \mathcal{P}$, then $w_i w_{i+1} \in I$: (the last arrow of w_i)$\cdot$(the first arrow of w_{i+1}) $\in I$,
- if $w_i, w_{i+1} \in \bar{\mathcal{P}}$, then $\bar{w}_{i+1} \bar{w}_i \in I$: $\overline{(\text{the first arrow of } w_{i+1})} \cdot \overline{(\text{the last arrow of } w_i)} \in I$,
- otherwise, $w_i w_{i+1}$ is reduced : (the last arrow of w_i) $\neq \overline{(\text{the first arrow of } w_{i+1})}$

We regard trivial paths e_v generalized strings as well.

Let us denote by $\widetilde{Gst}$ the set of generalized strings and define an equivalence relation $\sim_s$ on $\widetilde{Gst}$ as $w^1 \sim w^2$ iff $w^1 = w^2$ or $w^1 = \overline{w^2}$.

Then, define $Gst := \widetilde{Gst}/\sim_s$

Def) For a generalized string $w = w_1 \cdots w_n$, a <u>grading</u> μ is a sequence $(\mu(i))_{i=0,\dots,n}$ of integers defined by

$$\mu(0) := 0, \quad \mu(i) := \begin{cases} \mu(i-1) - 1 & \text{if } w_i \in \mathcal{P} \\ \mu(i-1) + 1 & \text{if } w_i \in \bar{\mathcal{P}} \end{cases}$$

Def) For a generalized string $w = w_1 \cdots w_n$, define a projective complex $(P_w^\bullet, d)$ as follows. Let $v_0 := s(w)$ and $v_i := t(w_i)$.

- $P_w^\nu := \bigoplus_{\mu(i) = \nu} P_{v_i}$
- δ is given by w_i : if $w_i \in \mathcal{P}$, it gives a differential $\delta_{w_i} : P_{t(w_i)} \longrightarrow P_{s(w_i)}$

  if $w_i \in \bar{\mathcal{P}}$, 〃 $\delta_{\bar{w}_i} : P_{s(w_i)} \longrightarrow P_{t(w_i)}$

Ex)
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & 4 \end{tikzcd}$$
$I = \{\alpha\gamma\}$

Reduced words : $\alpha \sim \bar{\alpha}$, $\beta \sim \bar{\beta}$, $\gamma \sim \bar{\gamma}$, $\alpha\beta \sim \bar{\beta}\bar{\alpha}$, $\alpha\gamma \sim \bar{\gamma}\bar{\alpha}$, $\bar{\beta}\gamma \sim \bar{\gamma}\beta$.

Gst with μ		Projective complexes (underlined component is at deg 0.)
e_1, e_2, e_3	$\longleftrightarrow$	$\underline{P_1}$, $\underline{P_2}$, $\underline{P_3}$
α, β, γ	$\longleftrightarrow$	$P_2 \xrightarrow{\alpha} \underline{P_1}$, $P_3 \xrightarrow{\beta} \underline{P_2}$, $P_4 \xrightarrow{\gamma} \underline{P_2}$
$\alpha\beta : (0,-1)$	$\longleftrightarrow$	$P_3 \xrightarrow{\alpha\beta} \underline{P_1}$
$\alpha/\gamma : (0,-1,-2)$	$\longleftrightarrow$	$P_4 \xrightarrow{\gamma} P_2 \xrightarrow{\alpha} \underline{P_1}$
$\bar{\beta}/\gamma : (0,1,0)$	$\longleftrightarrow$	$\underline{P_3 \oplus P_4} \longrightarrow P_2$

⑥ Generalized bands

Def) A <u>generalized band</u> is a generalized string $w = w_1 \cdots w_n$ such that

- $t(w) = s(w)$ and $w^2 = w_1 \cdots w_n w_1 \cdots w_n$ is a generalized string.
- w is not a power of other generalized string : $w \neq u^m$ for any u and $m \geq 2$.
- $\mu(n) = 0$ $\quad$: $\#\{ i : w_i \in \mathcal{P}\} = \#\{ i : w_i \in \overline{\mathcal{P}}\}$.

Let us denote by $\widetilde{GBa}$ the set of generalized bands and define an equivalence relation $\sim_r$ as

$$w^1 = w_1^1 \cdots w_n^1 \sim_r w^2 = w_1^2 \cdots w_n^2 \text{ iff } w^1 \sim_s w^2 \text{ or } w^1 \sim_s w^2[r] := w_{r+1}^2 \cdots w_n^2 w_1^2 \cdots w_r^2 .$$

Then, define $GBa := \widetilde{Gba}/\sim_r$.

Ex)
$$\begin{tikzcd} \bullet \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \bullet \arrow[r, "\gamma"] \arrow[dr, "\varepsilon"'] & \bullet \arrow[d, "\delta"] \\ & & \bullet \end{tikzcd}$$

$\alpha \cdot \beta^{-1}$ is a g.band, but $\alpha \cdot \beta^{-1} \cdot \alpha \cdot \beta^{-1}$ is not a g.band.

$\gamma \cdot \delta \cdot \varepsilon^{-1}$ is not a g.band ($\mu(2) = 1$)

Def) For a generalized band $w = w_1 \cdots w_n$, $\lambda \in \mathbb{K}^\times$, and $d \geq 1$, define a projective complex $(P^\bullet_{w,\lambda,d}, \delta)$ as follows.

- $P_w := \bigoplus_{\substack{\mu(i) = \nu \\ \nu < n}} P_{v_i} \otimes \mathbb{K}^d$
- δ is given by $w_1 \otimes id_{\mathbb{K}^d}, \ldots, w_{n-1} \otimes id_{\mathbb{K}^d}$, and $w_n \otimes J_{\lambda,d}$ where $J_{\lambda,d}$ is the Jordan block $\begin{bmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & 1 & \ddots & \\ & & & \lambda \end{bmatrix}_{d \times d}$

  that is, if $w_n \in \mathcal{P}$, it gives $P_{v_0} \otimes \mathbb{K}^d \xrightarrow{w_n \otimes J_{\lambda,d}} P_{v_{n-1}} \otimes \mathbb{K}^d$,

  if $w_n \in \overline{\mathcal{P}}$, it gives $P_{v_{n-1}} \otimes \mathbb{K}^d \xrightarrow{\overline{w_n} \otimes J_{\lambda,d}} P_{v_0} \otimes \mathbb{K}^d$.

Ex)
$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

$Gba = \{\alpha \cdot \beta^{-1}\}$

For $d = 1$, $\lambda \in \mathbb{K}^\times$, the corresponding complex is $P_2 \xrightarrow{\alpha + \lambda\beta} P_1$ =
$$\begin{tikzcd} 0 \arrow[r, shift left] \arrow[r, shift right] & \mathbb{K} \arrow[d, "{\begin{bmatrix} 1 \\ \lambda \end{bmatrix}}"] \\ \mathbb{K} \arrow[r, shift left, "{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}"] \arrow[r, shift right, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"'] & \mathbb{K}^2 \end{tikzcd} \simeq \begin{tikzcd} \mathbb{K} \arrow[r, shift left, "1"] \arrow[r, shift right, "-\lambda"'] & \mathbb{K} \end{tikzcd}$$

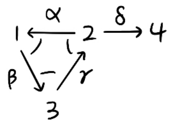
For general d and $\lambda \in \mathbb{K}^\times$, the corresponding complex is $P_2^d \xrightarrow{\alpha I_d + \beta J_{\lambda,d}} P_1^d$ =
$$\begin{tikzcd} 0 \arrow[r, shift left] \arrow[r, shift right] & \mathbb{K}^d \arrow[d, "{\begin{bmatrix} I_d \\ J_{\lambda,d} \end{bmatrix}}"] \\ \mathbb{K}^d \arrow[r, shift left, "{\begin{bmatrix} I_d \\ 0 \end{bmatrix}}"] \arrow[r, shift right, "{\begin{bmatrix} 0 \\ I_d \end{bmatrix}}"'] & \mathbb{K}^{2d} \end{tikzcd} \simeq \begin{tikzcd} \mathbb{K}^d \arrow[r, bend left, "I_d"] \arrow[r, bend right, "-J_{\lambda,d}"'] & \mathbb{K}^d \end{tikzcd}$$

ⓒ Non-perfect complexes

Rmk) When (Q, I) has a <u>forbidden cycle</u>, a cycle $\alpha_1 \cdots \alpha_n$ such that $\alpha_i\alpha_{i+1}, \alpha_n\alpha_1 \in I$, then $\mathbb{k}Q/I$ is not 'smooth.'

There is a complex $C^\cdot \in D^b(\mathbb{k}Q/I)$ which is not isomorphic to a bounded projective complex.

Ex)



Consider a (infinite) generalized string $w = \delta^{-1}/\alpha/\beta/\gamma/\alpha/\cdots$.

The corresponding complex is $\underbrace{\cdots \to P_3 \xrightarrow{\beta} P_1 \xrightarrow{\alpha} P_2 \xrightarrow{\gamma} P_3 \xrightarrow{\beta} P_1 \xrightarrow{\alpha} P_2 \xrightarrow{\gamma} P_3}_{\text{exact part}} \xrightarrow{\beta} P_1 \oplus P_4 \xrightarrow{\alpha \oplus \delta} P_2 \to 0$.

Def) A <u>left infinite generalized string</u> is a sequence $w = \cdots w_{-3} w_{-2} w_{-1}$ such that

- $w_{-k} \cdots w_{-1} \in \widehat{GSt}$ for any $k \geq 1$,
- there is some $k \geq 1$ such that $w_{-k}, w_{-k-1}, \cdots \in \overline{Q}_1$.

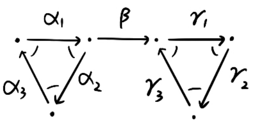
A <u>right infinite generalized string</u> is a sequence $w = w_1 w_2 w_3 \cdots$ such that

- $w_1 \cdots w_k \in \widehat{GSt}$ for any $k \geq 1$,
- there is some $k \geq 1$ such that $w_k, w_{k+1}, \cdots \in Q_1$.

An <u>infinite generalized string</u> is a sequence $\cdots w_{-2} w_{-1} w_0 w_1 w_2 \cdots$ such that

- $\cdots w_{-2} w_{-1} w_0$ is a left infinite generalized string,
- $w_0 w_1 w_2 \cdots$ is a right infinite generalized string.

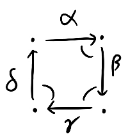
Ex)



$\cdots \overline{\gamma}_2 \overline{\gamma}_1 \overline{\gamma}_3 \overline{\gamma}_2 \overline{\gamma}_1 \overline{\beta} \alpha_2 \alpha_3$ is a left infinite string but $\cdots \alpha_2\alpha_3\alpha_1\alpha_2\alpha_3\alpha_1\gamma$ is not.

$\gamma_2 \gamma_3 \overline{\beta} \alpha_2 \alpha_3 \alpha_1 \alpha_2 \cdots$ is a right infinite generalized string.

$\cdots \overline{\gamma}_2 \gamma_1 \overline{\beta} \alpha_2 \alpha_3 \cdots$ is an infinite generalized string.



$\alpha/\beta/\gamma/\delta\alpha/\beta/\gamma/\delta\alpha/\cdots$ is not a right generalized string as $\delta\alpha \notin Q_1$.

Thm) [Bekkert-Merklen] Any indecomposable object in $D^b(\mathbb{k}Q/I)$ is isomorphic to one of

- $P^\cdot_w$ for some $w \in GSt$
- $P^\cdot_{w,\lambda,d}$ for some $w \in Gba$
- $P^\cdot_w$ for some (left/right) g. string.