§2. Differential graded categories

exact — earlier dg categories — now

Fix a commutative ring k

Definition: A differential graded category is a category enriched over $\Bbbk$-complexes $\mathcal{C}\Bbbk$

$\mathcal{C}\Bbbk$ has objects complexes $\dots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \to \dots$ $\forall n \in \mathbb{Z}$ V^n a $\Bbbk$-module and $d^{n+1} \circ d^n = 0$

Morphisms are $(f^n : V^n \to W^n)$ s.t. $d_W^n \circ f^n = f^{n+1} \circ d_V^n$ for all n

$\mathcal{C}\Bbbk$ comes with a monoidal structure, $V = (V^\bullet, d_V^\bullet), W = (W^\bullet, d_W) \in \mathcal{C}\Bbbk$

$V \otimes W$ is defined by $(V \otimes W)^n := \coprod_{n = p+q} V^p \otimes_k V^q$, $d^n_{V\otimes W}(v \otimes w) := d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w)$

$\rightsquigarrow$ Symmetric $V \otimes W \xrightarrow{\sim} W \otimes V$

$v \otimes w \longmapsto (-1)^{|w||v|} w \otimes v$ Koszul sign

Observation: $- \otimes V : \mathcal{C}\Bbbk \to \mathcal{C}\Bbbk$ has a right adjoint $\operatorname{Hom}^\bullet_\Bbbk(V, -)$, explicitly: Internal Hom

$\operatorname{Hom}^m_k(V, W) := \{ (f^n : V^n \to V^{n+m})_{m \in \mathbb{Z}} \}$

$$\begin{tikzcd}
\dots \arrow[r] & V^{n-1} \arrow[r] \arrow[d] & V^n \arrow[r] \arrow[d] & V^{n+1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & W^{n+m-1} \arrow[r] \arrow[ur, phantom, "?"] & W^{n+m} \arrow[r] \arrow[ur, phantom, "?"] & W^{n+m+1} \arrow[r] & \dots
\end{tikzcd}$$

⚠ No condition for the f^n

$\operatorname{Hom}^0_\Bbbk(V, W) \not\ni \mathcal{C}\Bbbk(V, W)$ in general

$d_{\operatorname{Hom}^\bullet_\Bbbk(V,W)}(f) := d_W \circ f - (-1)^{|f|} f \circ d_V$

For $V \in \mathcal{C}\Bbbk$ define $V[1]$ by $V[1]^n := V^{n+1}$, $d_{V[1]} := -d_V$ $\rightsquigarrow \operatorname{Hom}^n_\Bbbk(V, W) = \operatorname{Hom}^0_\Bbbk(V, W[n])$

Definition: A dg category $\mathcal{A}$ consits of a class of objects and for any two $X, Y \in \mathcal{A}$
$\mathcal{A}(X, Y) \in \mathcal{C}\Bbbk$ a complex of morphisms and composition morphisms which is associative + unital.

$$\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \longrightarrow \mathcal{A}(X, Z)$$ Leibnitz rule

Example:

(1) $\mathcal{C}_{dg}\Bbbk := (\mathcal{C}\Bbbk, \operatorname{Hom}^\bullet_\Bbbk(\ ,\ ))$ $\rightsquigarrow$ $d_V \in \operatorname{Hom}^1_\Bbbk(V, V)$ with d_V^2

$\rightsquigarrow d_{V\otimes W} = d_V \otimes 1 + 1 \otimes d_W$

⚠ $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$ 'switching g and x'

(2) B a $\Bbbk$-algebra, $\operatorname{Mod} B$ all (right) modules $\rightsquigarrow \mathcal{C}_{dg} B$ dg category of complexes of B modules

(3) dg $\Bbbk$-algebra A $\leftrightsquigarrow$ dg category with one object $*$, $A = \operatorname{Hom}_\mathcal{A}(*, *)$

(4) differential graded quiver, example:

$$\begin{tikzcd}
1 \arrow[r, "\alpha"'] \arrow[rr, bend left, "\gamma"] & 2 \arrow[r, "\beta"'] & 3
\end{tikzcd}$$

$\mathcal{A}(i, i) = \Bbbk\, \mathbb{1}_i$ $\mathcal{A}(1, 2) = \Bbbk \alpha$ $\mathcal{A}(2, 3) = \Bbbk \beta$

$\mathcal{A}(1, 3) = \underbrace{\Bbbk \gamma}_{-1} \oplus \underbrace{\Bbbk \beta\alpha}_{0}$ $d(\gamma) = \beta\alpha$

$|\alpha| = 0 = |\beta|$, $|\gamma| = -1$

$\rightsquigarrow$ dg path algebra / category

(5) $\mathcal{A}$ dg category $\rightsquigarrow$ opposite category $\mathcal{A}^{op}$ with $f \circ^{op} g := (-1)^{|f||g|} g \circ f$

(6) $\mathcal{A}, \mathcal{B}$ dg categories $\rightsquigarrow$ $\mathcal{A}\otimes\mathcal{B}$ with objects $\mathrm{Obj}\,\mathcal{A}\times\mathrm{Obj}\,\mathcal{B}$

$$\mathcal{A}\otimes\mathcal{B}\big((x,y),(x',y')\big) := \mathcal{A}(x,x')\otimes\mathcal{B}(y,y')$$

$$(g\otimes g')\circ(f\otimes f') = (-1)^{|g'||f|}(g\circ f)\otimes(g'\circ f')$$

(7)

$$\begin{array}{cccc} \otimes & 0 \longrightarrow 1 & & \mathcal{A} \\ & & & \\ \mathcal{B} & \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array} & \begin{array}{ccc} 00 & \longrightarrow & 01 \\ \downarrow & G & \downarrow \\ 10 & \longrightarrow & 11 \end{array} & \mathcal{A}\otimes\mathcal{B} \end{array}$$

Definition $\mathcal{A}, \mathcal{B}$ dg categories. A dg functor consists of a map $F\colon \mathrm{Obj}\,\mathcal{A}\to\mathrm{Obj}\,\mathcal{B}$ and

$\forall\, x, y \in \mathcal{A}$ $\quad F = F_{xy}\colon \mathcal{A}(x,y) \longrightarrow \mathcal{B}(Fx, Fy)$ morphism of complexes

compatible with composition and units. $\rightsquigarrow$ $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ category of dg functors (nat. transform ...)

Proposition $(\mathrm{dgCat}, \otimes)$ is a symmetric monoidal category with tensor unit $\Bbbk = *^{\circlearrowleft\Bbbk}$
Furthermore, it has an internal Hom $\mathrm{Fun}_{dg}(\mathcal{A},\mathcal{B})$

$\rightsquigarrow$ $\mathrm{Fun}_{dg}(\mathcal{A},\mathcal{B})$ dg category of dg functors $F\colon\mathcal{A}\longrightarrow\mathcal{B}$

Definition: $\mathcal{A}$ a dg category

- the 0-truncation of $\mathcal{A}$, $\tau_{\le 0}\mathcal{A}$, is the dg category with the same objects and for $x, y \in \tau_{\le 0}\mathcal{A}$
$$\tau_{\le 0}\mathcal{A}(x,y) = \big(\dots\to\mathcal{A}(x,y)^{-2}\to\mathcal{A}(x,y)^{-1}\to \underset{\ker(d^0)}{\overset{}{Z^0\mathcal{A}(x,y)}}\to\dots\big)$$
- the 0-cocycles of $\mathcal{A}$, $Z^0\mathcal{A}$, is the category with the same objects and for $x, y\in Z^0\mathcal{A}$
$$Z^0\mathcal{A}(x,y) := \ker\big(d^0_{\mathcal{A}(x,y)}\big)$$
- the 0-cohomology of $\mathcal{A}$, $H^0\mathcal{A}$, is the category with the same objects and for $x, y \in H^0\mathcal{A}$
$$H^0\mathcal{A}(x,y) := Z^0\mathcal{A}(x,y)\big/\mathrm{im}\big(d^{-1}_{\mathcal{A}(x,y)}\big)$$

Remark: We have dg functors $\quad H^0\mathcal{A} \twoheadleftarrow \tau_{\le 0}\mathcal{A} \hookrightarrow \mathcal{A}$

Definition: $F\colon\mathcal{A}\longrightarrow\mathcal{B}$ is a quasi-equivalence if $H^0F\colon H^0\mathcal{A}\longrightarrow H^0\mathcal{B}$ is an equivalence and for all $x,y\in\mathcal{A}$ $\quad F_{xy}$ is a quasi isomorphism. 'isomorphic in cohomology'

Examples:

(1) B a $\Bbbk$-algebra $\Rightarrow Z^0\mathcal{C}_{dg}B = \mathcal{C}B$, $\quad H^0\mathcal{C}_{dg}B = \mathcal{H}B$ homotopy category $\quad \mathcal{C}B/\text{homotopy}$

$\rightsquigarrow$ $f, g\colon X\longrightarrow Y$ in $\mathcal{C}B$ are homotopic if $\exists\, h\in\mathrm{Hom}^{-1}_B(X,Y)$ $\quad f-g = d_Y\circ h + h\circ d_X$

(2) $\mathcal{A} = A$ dg algebra $\Rightarrow$ $Z^0\mathcal{A} = Z^0A$, $\quad H^0\mathcal{A} = H^0A$

(3) $$\mathcal{A} = \begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\gamma"'] & 2 \arrow[r, "\beta"] & 3 \end{tikzcd} \qquad d(\gamma) = \beta\alpha$$ "sequence upto homotopy"

$$Z^0\mathcal{A} = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad\qquad H^0\mathcal{A} = 1\xrightarrow{\alpha} 2\xrightarrow{\beta} 3$$

Reminder: B a $\Bbbk$-algebra $\rightsquigarrow$ $\mathcal{B} = B$ as category with one object

$$\Rightarrow \mathrm{Mod}\,B \simeq \mathrm{Fun}(\mathcal{B}^{op}, \mathrm{Mod}\,\Bbbk)$$
right modules

$$\Rightarrow \mathcal{C}B \simeq \mathrm{Fun}(\mathcal{B}^{op}, \mathcal{C}\Bbbk) = \mathrm{Fun}(\mathcal{B}^{op}, \mathcal{C}_{dg}\Bbbk)$$

Definition: $\mathcal{A}$ a dg category

$\mathcal{C}\mathcal{A} := \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{C}_{dg}\Bbbk)$ category of (right) $\mathcal{A}$-modules

Morphisms = natural transformations

$M, N \in \mathcal{C}\mathcal{A}$ $\quad f: M \longrightarrow N$, f_X morphism of complexes

$$\forall\ X \xrightarrow{\varphi} Y \qquad \begin{tikzcd} MX \arrow[d, "f_X"'] \arrow[dr, phantom, "\circlearrowleft"] & MY \arrow[l, "M\varphi"'] \arrow[d, "f_Y"] \\ NX & NY \arrow[l] \end{tikzcd}$$

Example: $X \in \mathcal{A} \implies X^\wedge := \mathcal{A}(-, X) : \mathcal{A}^{op} \longrightarrow \mathcal{C}\Bbbk$ the representable dg module

Yoneda Lemma $\mathcal{A}$ dg cat, $X \in \mathcal{A}$, $M \in \mathcal{C}\mathcal{A} \implies$

$$\mathcal{C}\mathcal{A}(X^\wedge, M) \xrightarrow{\sim} Z^0 MX \qquad \text{morphisms!}$$
$$\varphi \longmapsto \varphi(1_X)$$

$\rightsquigarrow$ Solution: dg-category of dg modules

Definition: $\mathcal{A}$ a dg-category, define the dg cat of $\mathcal{A}$ modules $\mathcal{C}_{dg}\mathcal{A} := \mathrm{Fun}_{dg}(\mathcal{A}^{op}, \mathcal{C}_{dg}\Bbbk)$,
Explicitly for
$M, N \in \mathcal{C}_{dg}\mathcal{A}$

$$\mathcal{C}_{dg}\mathcal{A}(M,N)^n := \left\{ f \in \prod_{X \in \mathcal{A}} \mathrm{Hom}^n_{\Bbbk}(MX, NX) \;\middle|\; \forall m\ \forall \varphi \in \mathcal{A}(X,Y)^m \begin{tikzcd} MX \arrow[d, "f_X"'] \arrow[dr, phantom, "(-1)^{n\cdot m}"] & MY \arrow[l, "\varphi"'] \arrow[d, "f_Y"] \\ NX & NY \arrow[l] \end{tikzcd} \right\}$$

no constrains ⚠

Remark: $\mathcal{C}\mathcal{A} = Z^0 \mathcal{C}_{dg}\mathcal{A}$

Yoneda Lemma $\mathcal{A}$ dg cat, $X \in \mathcal{A}$, $M \in \mathcal{C}_{dg}\mathcal{A} \implies$

$$\mathcal{C}_{dg}\mathcal{A}(X^\wedge, M) \xrightarrow{\sim} MX$$
$$\varphi \longmapsto \varphi(1_X)$$

Corollary: Let $\mathcal{H}\mathcal{A} := H^0 \mathcal{C}_{dg}\mathcal{A}$, then $\mathcal{H}\mathcal{A}(X^\wedge, M) \cong H^0 MX$

Recall: $g: M \longrightarrow N$ is a quasi isomorphism if for all $X \in \mathcal{A}$ $g_X : MX \longrightarrow NX$ is a qiso, i.e.

$$H^0(g[n]) : H^0(MX[n]) \longrightarrow H^0(NX[n]) \text{ is an isomorphism}$$ (invariant under homotopy)

Definition: $\mathcal{A}$ a dg category. The derived category of $\mathcal{A}$ is the localisation of $\mathcal{H}\mathcal{A}$ at the qisos

$$\mathcal{D}\mathcal{A} := \mathcal{H}\mathcal{A}[\text{qiso}^{-1}]$$

$\rightsquigarrow$ Universal Property

$$\begin{tikzcd} \mathcal{H}\mathcal{A} \arrow[d, "\text{can}"'] \arrow[dr, "F \text{ s.t. } F(\text{qisos}) \in \text{isos}"] & \\ \mathcal{D}\mathcal{A} \arrow[r, dotted, "\exists! F"'] & \mathcal{C} \end{tikzcd}$$

Theorem The localisation functor $\text{can}: \mathcal{H}\mathcal{A} \longrightarrow \mathcal{D}\mathcal{A}$ admits two fully faithful adjoints

$$\begin{tikzcd} \mathcal{H}\mathcal{A} \arrow[r, "\text{can}"] & \mathcal{D}\mathcal{A} \arrow[l, bend right, "p"', "\perp"] \arrow[l, bend left, "i", "\perp"'] \end{tikzcd}$$

"projective resolution"

"injective resolution"

$\implies$ In particular, $\mathcal{D}\mathcal{A}$ is indeed a category.

Reminder: $\mathcal{A}$ ~~abelian~~/exact dg cat $\rightsquigarrow$ $\mathcal{C}\mathcal{A}$ has an exact structure given by degreewise split s.e.s.

In fact, this is a Frobenius exact structure

$\implies$ $\mathcal{H}\mathcal{A} = \underline{\mathcal{C}\mathcal{A}}$ stable category canonically carries the structure of a triangulated category

projective-injectives are the null homotopic complexes,

$f: X \to Y$ null homotopic iff X factors through $I(X)$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & I(X) \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\ Y \arrow[r] & \mathrm{Cone}(f) \arrow[r] & X[1] \end{tikzcd}$$

'injective hull'

exact / distinguished triangle

⇒ If $\mathcal{D}\mathcal{A}$ is a category is canonically triangulated since

$$\mathcal{D}\mathcal{A} = \mathcal{H}\mathcal{A}[qis^{-1}] \cong \mathcal{H}\mathcal{A}/\{X \underset{qis}{\simeq} 0\}$$ canonically triangulated since $\{X \underset{qis}{\simeq} 0\}$ is a <u>null system</u>

<u>Definition:</u> A dg category $\mathcal{A}$ is called pretriangulated if the fully faithfull functor

$$h := H^0 y : H^0\mathcal{A} \longrightarrow \mathcal{H}\mathcal{A}$$ (induced by dg Yoneda functor $y : \mathcal{A} \longrightarrow \mathcal{C}_{dg}\mathcal{A}$)

induces the structure of a triangulated category on $H^0\mathcal{A}$.
Explicitly h is supposed to be stable under the surpension $[\pm 1]$ and taking cones.

In this case we call the triangulated category $H^0\mathcal{A}$ <u>algebraic triangulated category</u>.

<u>Theorem</u> $\mathcal{T}$ algebraic triangulated category iff $\mathcal{T}$ stable module category of some Frobenius exact category.

<u>Lemma</u> The Verdier Localisation of an algebraic category is algebraic. (Ignoring set theoretic problems)

<u>Remark:</u> 'all triangulated categories in algebra are algebraic'

<u>Outlook:</u>

$$\begin{tikzcd}
\text{pretriangulated dg category} \arrow[r, hook, "\text{generalise}"'] \arrow[d, "H^0"] & \boxed{\text{exact dg category}} \arrow[d, "H^0"] & & \text{[Xiaofa Chen 2023]} \\
\text{algebraic triangulated cat} \arrow[r, hook, "\text{generalise}"'] & \text{algebraic extriangulated cat} & & \text{[Nakaoka-Palu 2019]} \\
\text{abelian cat} \arrow[u, hook] \arrow[r, hook] \arrow[uu, bend left] & \text{exact categories} \arrow[u, hook] \arrow[uu, bend right]
\end{tikzcd}$$