CHARMS Summer School in Versailles (May 2024) — Lecture notes by Daniel Perniok

Additive, abelian and exact categories

> **Reference:**
> B. Keller: Minicourse "Derived categories of exact categories", March 1–5, 2021

1. Additive categories

Def: An *additive category* is a category $\mathcal{A}$ such that

(Add 1) $\mathcal{A}$ has a zero object 0 (i.e. an object that is both initial and terminal)

(Add 2) $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ has an abelian group structure such that composition is biadditive.

(Add 3) $\mathcal{A}$ has biproducts, i.e. $\forall\, X_1, X_2 \in \mathrm{Ob}(\mathcal{A})$ $\exists$ object $X =: X_1 \oplus X_2$ and morphisms

$$\begin{tikzcd} X_1 \arrow[r, shift right, "\sigma_1"'] & X \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & X_2 \arrow[l, shift left, "\sigma_2"] \end{tikzcd}$$

such that
- $\pi_i \sigma_j = \delta_{ij}\, \mathrm{id}_{X_i}$
- $\sigma_1 \pi_1 + \sigma_2 \pi_2 = \mathrm{id}_X$

Rmk (1) The zero element in the abelian group $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ equals the unique morphism

$$\begin{tikzcd} X \arrow[rr, "0"] \arrow[dr] & & Y \\ & 0 \arrow[ur] & \end{tikzcd}$$

(2) (X, σ_1, σ_2) is a coproduct and (X, π_1, π_2) is a product of X_1 and X_2

(3) The group structures on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ are intrinsic and no additional data!

Given $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$

$$\begin{tikzcd} X \arrow[rr, "f+g"] \arrow[d, "\begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix} = \sigma_1^X + \sigma_2^X"'] & & Y \\ X \oplus X \arrow[rr, "\sigma_1^Y f \pi_1^X + \sigma_2^Y g \pi_2^X"', "= \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}" below=12pt] & & Y \oplus Y \arrow[u, "\pi_1^Y + \pi_2^Y = \begin{pmatrix} \mathrm{id}_Y & \mathrm{id}_Y \end{pmatrix}"'] \end{tikzcd}$$

The black morphisms can be constructed using only universal properties of (co)products and the fact that we have a zero object (and hence zero morphisms)

2. Abelian categories

Def Let $\mathcal{A}$ be a category with a zero object and $X \xrightarrow{f} Y$ a morphism.

- $(K, K \xrightarrow{i} X)$ is a kernel of f if ($fi = 0$) $\forall\; T \xrightarrow{t} X$ with $ft = 0$ $\exists!\; T \xrightarrow{\tilde{t}} K$ with $i\tilde{t} = t$

$$\begin{tikzcd} & T \arrow[dl, dashed, "\exists \tilde{t}"'] \arrow[d, "\forall t"'] \arrow[dr, "0"] & \\ K \arrow[r, "i"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

- $(C, Y \xrightarrow{p} C)$ is a cokernel of f if $pf = 0$ and $\forall\; Y \xrightarrow{t} T$ with $tf = 0$ $\exists!\; C \xrightarrow{\tilde{t}} T$ with $\tilde{t}p = t$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "0"'] & Y \arrow[r, "p"] \arrow[d, "\forall t"] & C \arrow[dl, dashed, "\exists \tilde{t}"] \\ & T & \end{tikzcd}$$

- The image of f is $\mathrm{Im}(f) := \ker(Y \longrightarrow \mathrm{Cok}(f))$

  The coimage of f is $\mathrm{Coim}(f) := \mathrm{Cok}(\ker(f) \longrightarrow X)$

$\rightsquigarrow$ We get an induced morphism $\bar{f} : \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$

$$\begin{tikzcd} K \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "p"] & C \\ & \mathrm{Coim}(f) \arrow[r, dashed, "\bar{f}"] & \mathrm{Im}(f) \arrow[u] & \end{tikzcd}$$

Def: An abelian category is an additive category $\mathcal{A}$ such that every morphism has a kernel and cokernel and for every morphism $f : X \longrightarrow Y$ the induced morphism $\bar{f} : \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$ is an isomorphism.

Examples of abelian categories:

- $\mathrm{Mod}\, R$ for a ring R
- $\mathrm{Coh}(X)$ for a scheme X
- $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ for a small cat. $\mathcal{C}$ and an abelian cat. $\mathcal{A}$

Non-example: $\mathcal{A}$ = category of f.g. free abelian groups, then $\mathcal{A}$ has all kernels and cokernels and $\mathrm{Im}(f) \cong \mathrm{Coim}(f)$ for every morphism f, but $\mathcal{A}$ is not abelian.

3. Exact categories

Def: An *exact category* is an additive category $\mathcal{A}$ together with a class of kernel-cokernel pairs (i,p), called *conflations*, s.t.

inflation $\nearrow$ i, p $\nwarrow$ deflation

(Ex 0) id_0 is a deflation

(Ex 1) The composition of two deflations is a deflation

(Ex 2) For every deflation p and morphism f there exists a pullback and p' is again a deflation

$$\begin{tikzcd} Y' \arrow[r, two heads, "p'"] \arrow[d] & Z' \arrow[d, "f"] \\ Y \arrow[r, two heads, "p"] & Z \end{tikzcd}$$

(Ex 2^{op}) For every inflation i and morphism f there exists a pushout and i' is again an inflation

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "f"'] & Y \arrow[d] \\ X' \arrow[r, tail, "i'"] & Y' \end{tikzcd}$$

Recall:

$$\begin{tikzcd} T \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] \arrow[dr, dashed, "\exists! \gamma"] & & \\ & W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

is a *pullback* square if

$\forall$ $T \xrightarrow{\alpha} X$, $T \xrightarrow{\beta} Y$ with $f\alpha = g\beta$

$\exists!$ $\gamma : T \longrightarrow W$ such that $\begin{cases} g'\gamma = \alpha \\ f'\gamma = \beta \end{cases}$

dual notion: *pushout* square [...]

Rmk (1) The dual statements (Ex 0^{op}) and (Ex 1^{op}) can be derived from the above axioms.

(2) For every isomorphism φ, the diagram is a pullback square

$$\begin{tikzcd} X \arrow[r, "\varphi", "\cong"'] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r, "\mathrm{id}_0"] & 0 \end{tikzcd}$$

$\Longrightarrow$ isomorphisms are deflations (and also inflations by the dual argument)

(3) For all $X, Z \in \mathrm{Ob}(\mathcal{A})$, $X \xrightarrow{0} 0$ is a deflation and

$$\begin{tikzcd} X \oplus Z \arrow[r, "(0\ \mathrm{id}_Z)"] \arrow[d, "(\mathrm{id}_X\ 0)"'] & Z \arrow[d, "0"] \\ X \arrow[r, "0"] & 0 \end{tikzcd}$$

is a pullback square

$$\Longrightarrow \quad X \xrightarrow{\begin{pmatrix} \mathrm{id}_X \\ 0 \end{pmatrix}} X \oplus Z \xrightarrow{(0\ \mathrm{id}_Z)} Z \qquad (*)$$

is a conflation

<u>Examples</u> (1) Every additive category $\mathcal{A}$ has an exact structure given by $\{\text{conflations}\} = \left\{ \begin{array}{l} \text{kernel-cokernel pairs} \\ \text{isomorphic to } (*) \end{array} \right\}$

(2) An abelian category can have different exact structures
(e.g. conflations = $\underbrace{\text{all s.e.s}}_{\text{i.e. all kernel-cokernel pairs}}$, conflations = split s.e.s., ...)

(3) $\mathcal{A}$ abelian, $\mathcal{B} \subseteq \mathcal{A}$ full, extension-closed subcategory with conflations = all s.e.s. with objects in $\mathcal{B}$

$\Longrightarrow$ $\mathcal{B}$ is an exact category
(moreover every small exact category is of this form)

(4) I small category, $\mathcal{A}$ exact category, then the functor category $\mathrm{Fun}(I, \mathcal{A})$ is exact with

conflation = pointwise conflations

4. The derived category of an exact category

Let $\mathcal{A}$ be an exact category.

We write $C(\mathcal{A})$ for the category of chain complexes

$$\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots \qquad (d^i d^{i-1} = 0)$$

A morphism of chain complexes $f^\bullet = (f^i : X^i \longrightarrow Y^i)_{i \in \mathbb{Z}}$

satisfies $f^{i+1} d_X^i = d_Y^i f^i$.

$f^\bullet$ is nullhomotopic if there are morphisms $s^i : X^i \longrightarrow Y^{i-1}$ in $\mathcal{A}$

such that $\forall i \in \mathbb{Z}$

$$f^i = s^{i+1} d_X^i + d_Y^{i-1} s^i$$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[r, "d_X^{i-1}"] \arrow[d, "f^{i-1}"] & X^i \arrow[r, "d_X^i"] \arrow[d, "f^i"] \arrow[dl, "s^i"'] & X^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] \arrow[dl, "s^{i+1}"'] & \cdots \\
\cdots \arrow[r] & Y^{i-1} \arrow[r, "d_Y^{i-1}"] & Y^i \arrow[r, "d_Y^i"] & Y^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Def. The homotopy category of (the underlying additive category of) $\mathcal{A}$ is $H(\mathcal{A}) := C(\mathcal{A}) / \left(\begin{smallmatrix}\text{nullhomotopic} \\ \text{morphisms}\end{smallmatrix}\right)$, i.e.

$$\mathrm{Ob}(H(\mathcal{A})) = \mathrm{Ob}(C(\mathcal{A})), \qquad \mathrm{Hom}_{H(\mathcal{A})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet) / \left(\begin{smallmatrix}\text{nullhomotopic} \\ \text{morphisms} \\ X^\bullet \longrightarrow Y^\bullet\end{smallmatrix}\right)$$

Rmk For an additive category $\mathcal{A}$, $C(\mathcal{A})$ is an exact category with

conflation = levelwise split s.e.s.

Moreover $C(\mathcal{A})$ is a Frobenius category with

$\{\text{projectives}\} = \{\text{injectives}\} = \{\underbrace{\text{contractible}}_{\text{i.e. id is nullhomotopic}} \text{ complexes}\}$

and the homotopy category $H(\mathcal{A})$ coincides with $\underline{C(\mathcal{A})}$

(stable category of a Frobenius category)

From now on, we will also use the exact structure on $\mathcal{A}$ which did not play a role so far.

<u>Def:</u> A complex $X^\bullet \in C(\mathcal{A})$ is *acyclic* if there are factorizations

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[rr, "d^{i-1}"] \arrow[dr] & & X^i \arrow[rr, "d^i"] \arrow[dr] & & X^{i+1} \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[ur] & & Z^i \arrow[ur] & & Z^{i+1} \arrow[ur] & & \cdots
\end{tikzcd}$$

such that $Z^i \longrightarrow X^i \longrightarrow Z^{i+1}$ is a conflation in $\mathcal{A}$ $\forall i \in \mathbb{Z}$.

<u>Def:</u> A morphism $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ in $C(\mathcal{A})$ is a *quasi-isomorphism* if the *mapping cone*

$$\operatorname{cone}(f^\bullet) := \quad \cdots \longrightarrow \underset{i}{Y^i \oplus X^{i+1}} \xrightarrow{\begin{pmatrix} d_Y^i & f^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix}} \underset{i+1}{Y^{i+1} \oplus X^{i+2}} \longrightarrow \cdots$$

is $\underbrace{\text{isomorphic in } H(\mathcal{A})}_{\text{"homotopy equivalent"}}$ to an acyclic complex.

<u>Example</u> Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$, then

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism.

In particular

$$\operatorname{cone}\left(\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd} \right)$$

$$= \left(\cdots \longrightarrow 0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow \cdots \right)$$

is quasi-isomorphic to $\left(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Z \longrightarrow 0 \longrightarrow \cdots \right)$

("conflations in $\mathcal{A}$ give rise to distinguished triangles in $D(\mathcal{A})$")

Def: The *derived category* of the exact category $\mathcal{A}$ is the localization

$$D(\mathcal{A}) := H(\mathcal{A})[(\text{classes of quasi-isomorphisms})^{-1}]$$
$$= C(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}]$$

Recall (Universal property of the localization of a category)

Let $\mathcal{C}$ be a category, $W \subseteq \mathrm{Mor}(\mathcal{C})$ a class of morphisms.

Then $\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$ is a *localization* of $\mathcal{C}$ at W if

- γ maps W to isomorphisms
- if $F: \mathcal{C} \longrightarrow \mathcal{D}$ is any functor which maps W to isomorphisms, then there exists a unique functor $\tilde{F}$ such that $\tilde{F} \circ \gamma = F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "\forall F"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C}[W^{-1}] \arrow[dl, dashed, "\exists! \tilde{F}"] \\ \mathcal{D} & \end{tikzcd}$$

Rmk:

"hard" (no calculus of fractions)

$$\begin{tikzcd} C(\mathcal{A}) \arrow[rr, "\text{localization at quasi-isomorphisms}"] \arrow[dr, "\text{factor out nullhomotopic morphisms}"'] & & D(\mathcal{A}) \\ & H(\mathcal{A}) \arrow[ur, "\text{localization at (classes of) quasi-isomorphisms}"'] & \end{tikzcd}$$

"easy"

can be made explicit!
I.e. we have a calculus of fractions because we benefit from the structure of a triangulated category on $H(\mathcal{A})$)

Reference: A. Yekutieli: "Derived categories" chapter 6+7

Addendum: Concrete construction of $D(\mathcal{A})$

Let $\mathcal{C} := H(\mathcal{A})$ and $W := \{\text{classes of quasi-isomorphisms}\} \subseteq \mathrm{Mor}(\mathcal{C})$

Lemma (1) W is closed under composition

(2) $\forall\, f \in \mathrm{Mor}(\mathcal{C}),\ s \in W\ \exists\, g \in \mathrm{Mor}(\mathcal{C}),\ t \in W$

such that $gs = tf$ $\quad$ (" $fs^{-1} = t^{-1}g$ ")

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "f"] & \\ Y \arrow[dr, dashed, "g"'] & & Z \arrow[dl, dashed, "t"] \\ & V & \end{tikzcd}$$

(3) $\forall\, s \in W,\ f \in \mathrm{Mor}(\mathcal{C})$

such that $fs = 0$

$\exists\, t \in W$ with $tf = 0$

$$\begin{tikzcd} X' \arrow[r, "s"] \arrow[dr, "0"'] & X \arrow[d, "f"] \arrow[dr, "0"] & \\ & Y \arrow[r, "t"] & Y' \end{tikzcd}$$

proof One has to use that

$$\mathcal{N} := \left\{ \begin{array}{l} \text{complexes which are homotopy-} \\ \text{equivalent to an acyclic complex} \end{array} \right\} \subseteq \mathcal{C} = H(\mathcal{A})$$

is a triangulated subcategory, e.g. (1) is then a direct consequence of the octahedral axiom. [...] ∎

Construction / Proposition: The following is a localization of $\mathcal{C}$ at W:

* $\mathrm{Ob}(\mathcal{C}[W^{-1}]) = \mathrm{Ob}(\mathcal{C})$

* $$\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X,Y) := \left\{ \underbrace{(f,s)}_{s^{-1}f\, =\, \substack{\searrow f \\ s \nwarrow}} \ \middle|\ \begin{tikzcd}[sep=small] X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\ & Z & \end{tikzcd} \ \begin{array}{l}\text{morphisms in } \mathcal{C} \\ \text{and } s \in W\end{array} \right\} \Big/ \sim$$

with $(f_1,s_1) \sim (f_2,s_2)$ $:\Longleftrightarrow$ $\exists$ diagram

$$\begin{tikzcd} X \arrow[d, "f_1"'] \arrow[drr, "f_2" near start] & & Y \arrow[dll, "s_1"' near start] \arrow[d, "s_2"] \\ Z_1 \arrow[dr, dashed, "a"'] & & Z_2 \arrow[dl, dashed, "b"] \\ & Z & \end{tikzcd}$$

with $bs_2 \in W$

* composition via part (2) of the lemma

* $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ $\quad$ on objects: $X \longmapsto X$

  on morphisms: $f \longmapsto (f, \mathrm{id}) = \substack{\searrow f \\ \mathrm{id} \nwarrow}$