

Additive, abelian and exact categories

Reference:

B. Keller: Minicourse "Derived categories of exact categories",
March 1-5, 20211. Additive categoriesDef: An additive category is a category \mathcal{A} such that(Add1) \mathcal{A} has a zero object 0 (i.e. an object that is both initial and terminal)(Add2) $\text{Hom}_{\mathcal{A}}(X, Y)$ has an abelian group structure such that composition is biadditive.(Add3) \mathcal{A} has biproducts, i.e. $\forall X_1, X_2 \in \text{Ob}(\mathcal{A}) \exists$ object $X = X_1 \oplus X_2$ and morphisms

$$X_1 \begin{array}{c} \xleftarrow{\pi_1} \\ \xrightarrow{\sigma_1} \end{array} X \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{\sigma_2} \end{array} X_2$$

such that

$$\bullet \pi_i \sigma_j = \delta_{ij} \text{id}_{X_i}$$

$$\bullet \sigma_1 \pi_1 + \sigma_2 \pi_2 = \text{id}_X$$

Prop (1) The zero element in the abelian group $\text{Hom}_{\mathcal{A}}(X, Y)$ equals the unique morphism

$$X \begin{array}{c} \xrightarrow{0} \\ \searrow 0 \\ \nearrow 0 \end{array} Y$$

(2) (X, σ_1, σ_2) is a coproduct and (X, π_1, π_2) is a product of X_1 and X_2 (3) The group structures on $\text{Hom}_{\mathcal{A}}(X, Y)$ are intrinsic and no additional data!

Given $X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$

$$\begin{array}{ccc} X & \xrightarrow{f+g} & Y \\ \downarrow \begin{pmatrix} \text{id}_X \\ \text{id}_X \end{pmatrix} & & \uparrow \begin{pmatrix} \pi_1^Y + \pi_2^Y \\ \text{id}_Y \end{pmatrix} \\ X \oplus X & \xrightarrow{\begin{pmatrix} \sigma_1^Y f \pi_1^X + \sigma_2^Y g \pi_2^X \\ \parallel \\ \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \end{pmatrix}} & Y \oplus Y \end{array}$$

The black morphisms can be constructed using only universal properties of (co)products and the fact that we have a zero object (and hence zero morphisms)

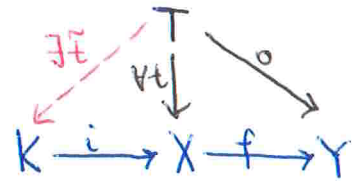
2. Abelian categories

Def Let \mathcal{A} be a category with a zero object and $X \xrightarrow{f} Y$ a morphism.

• $(K, K \xrightarrow{i} X)$ is a kernel of f

if $\forall T \xrightarrow{t} X$ with $ft=0$

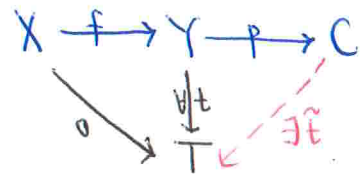
$\exists! T \xrightarrow{\tilde{t}} K$ with $i\tilde{t}=t$



• $(C, Y \xrightarrow{p} C)$ is a cokernel of f if

$pf=0$ and $\forall Y \xrightarrow{p} T$ with $tf=0$

$\exists! C \xrightarrow{\tilde{p}} T$ with $\tilde{p}p=t$

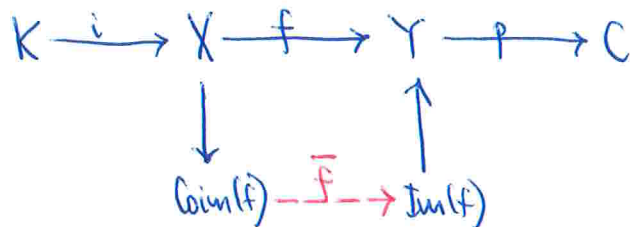


• The image of f is $\text{Im}(f) := \ker(Y \rightarrow \text{Cok}(f))$

The coimage of f is $\text{Coim}(f) := \text{Cok}(\ker(f) \rightarrow X)$

\rightarrow We get an induced morphism

$\bar{f}: \text{Coim}(f) \rightarrow \text{Im}(f)$



Def: An abelian category is an additive category \mathcal{A} such that every morphism has a kernel and cokernel and for every morphism $f: X \rightarrow Y$ the induced morphism $\bar{f}: \text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism.

Examples of abelian categories:

- $\text{Mod } R$ for a ring R
- $\text{Coh}(X)$ for a scheme X
- $\text{Fun}(\mathcal{C}, \mathcal{A})$ for a small cat. \mathcal{C} and an abelian cat. \mathcal{A}

Non-example: $\mathcal{A} =$ category of f.g. free abelian groups, then \mathcal{A} has all kernels and cokernels and $\text{Im}(f) \cong \text{Coim}(f)$ for every morphism f , but \mathcal{A} is not abelian.

3. Exact categories

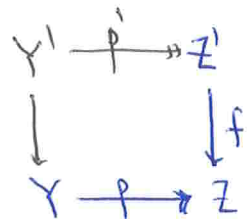
Def: An exact category is an additive category \mathcal{A} together with a class of kernel-kernel pairs (i, p) , called conflations, s.t.

inflation \nearrow \nwarrow deflation

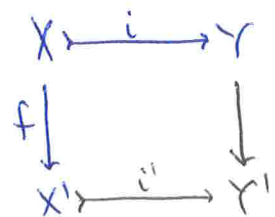
(Ex 0) id_0 is a deflation

(Ex 1) The composition of two deflations is a deflation

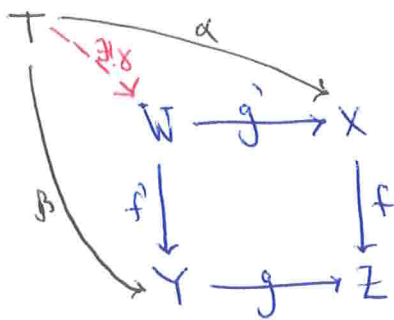
(Ex 2) For every deflation p and morphism f there exists a pullback and p' is again a deflation



(Ex 2^{op}) For every inflation i and morphism f there exists a pushout and i' is again an inflation



Recall:



is a pullback square if

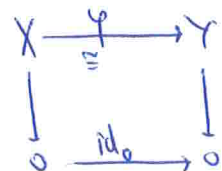
$$\forall \begin{array}{ccc} T & \xrightarrow{\alpha} & X \\ T & \xrightarrow{\beta} & Y \end{array} \text{ with } f\alpha = g\beta$$

$$\exists! \gamma: T \rightarrow W \text{ such that } \begin{cases} g'\gamma = \alpha \\ f'\gamma = \beta \end{cases}$$

dual notion: pushout square [...]

Remark (1) The dual statements (Ex 0^{op}) and (Ex 1^{op}) can be derived from the above axioms.

(2) For every isomorphism φ , the diagram is a pullback square



\Rightarrow isomorphisms are deflations
(and also inflations by the dual argument)

(3) For all $X, Z \in \text{Obj}(\mathcal{A})$, $X \xrightarrow{0} 0$ is a deflation and

$$\begin{array}{ccc}
 X \oplus Z & \xrightarrow{(0 \text{ id}_Z)} & Z \\
 \downarrow (\text{id}_X \ 0) & & \downarrow 0 \\
 X & \xrightarrow{0} & 0
 \end{array}
 \text{ is a pullback square}$$

$$\Rightarrow X \xrightarrow{\begin{pmatrix} \text{id}_X \\ 0 \end{pmatrix}} X \oplus Z \xrightarrow{(0 \text{ id}_Z)} Z \quad \textcircled{*}$$

is a conflation

Examples

(1) Every additive category \mathcal{A} has an exact structure given by $\{\text{conflations}\} = \left\{ \begin{array}{l} \text{kernel-cokernel pairs} \\ \text{isomorphic to } \textcircled{*} \end{array} \right\}$

(2) An abelian category can have different exact structures
 (e.g. $\text{conflations} = \underbrace{\text{all s.e.s.}}_{\text{i.e. all kernel-cokernel pairs}}$, $\text{conflations} = \text{split s.e.s., ...}$)

(3) \mathcal{A} abelian, $\mathcal{B} \subseteq \mathcal{A}$ full, extension-closed subcategory with
 $\text{conflations} = \text{all s.e.s. with objects in } \mathcal{B}$

$\Rightarrow \mathcal{B}$ is an exact category
 (moreover every small exact category is of this form)

(4) I small category, \mathcal{A} exact category, then the functor category

$\text{Fun}(I, \mathcal{A})$ is exact with

$\text{conflation} = \text{pointwise conflations}$

4. The derived category of an exact category

Let \mathcal{A} be an exact category.

We write $C(\mathcal{A})$ for the category of chain complexes

$$\dots \rightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \rightarrow \dots \quad (d^i d^{i-1} = 0)$$

A morphism of chain complexes $f^\bullet = (f^i: X^i \rightarrow Y^i)_{i \in \mathbb{Z}}$

satisfies $f^{i+1} d_x^i = d_y^i f^i$.

f^\bullet is nullhomotopic if there are morphisms $s^i: X^i \rightarrow Y^{i-1}$ in \mathcal{A}

such that $\forall i \in \mathbb{Z}$

$$f^i = s^{i+1} d_x^i + d_y^{i-1} s^i$$

$$\begin{array}{ccccccc} \dots & \rightarrow & X^{i-1} & \xrightarrow{d_x^{i-1}} & X^i & \xrightarrow{d_x^i} & X^{i+1} & \rightarrow \dots \\ & & \downarrow f^{i-1} & \swarrow s^i & \downarrow f^i & \swarrow s^{i+1} & \downarrow f^{i+1} & \\ \dots & \rightarrow & Y^{i-1} & \xrightarrow{d_y^{i-1}} & Y^i & \xrightarrow{d_y^i} & Y^{i+1} & \rightarrow \dots \end{array}$$

Def. The homotopy category of (the underlying additive category of)

\mathcal{A} is $H(\mathcal{A}) := C(\mathcal{A}) / \left(\begin{array}{l} \text{nullhomotopic} \\ \text{morphisms} \end{array} \right)$, i.e.

$$\text{Ob}(H(\mathcal{A})) = \text{Ob}(C(\mathcal{A})) \quad , \quad \text{Hom}_{H(\mathcal{A})}(X^\bullet, Y^\bullet) = \text{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet) / \left(\begin{array}{l} \text{nullhomotopic} \\ \text{morphisms} \\ X^\bullet \rightarrow Y^\bullet \end{array} \right)$$

Prop. For an additive category \mathcal{A} , $C(\mathcal{A})$ is an exact category with

conflation = levelwise split s.e.s.

Moreover $C(\mathcal{A})$ is a Frobenius category with

$$\{ \text{projectives} \} = \{ \text{injectives} \} = \{ \text{contractible complexes} \}$$

i.e. id is nullhomotopic

and the homotopy category $H(\mathcal{A})$ coincides with $C(\mathcal{A})$

(stable category of a Frobenius category)

From now on, we will also use the exact structure on \mathcal{A} which did not play a role so far.

Def: A complex $X^\bullet \in C(\mathcal{A})$ is acyclic if there are

factorizations

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^{i-1} & \xrightarrow{d^{i-1}} & X^i & \xrightarrow{d^i} & X^{i+1} & \longrightarrow \cdots \\ & & \nearrow & & \searrow & & \nearrow & \\ & & & & Z^i & & Z^{i+1} & \end{array}$$

such that $Z^i \longrightarrow X^i \longrightarrow Z^{i+1}$ is a conflation in \mathcal{A} $\forall i \in \mathbb{Z}$.

Def: A morphism $f^\bullet: X^\bullet \longrightarrow Y^\bullet$ in $C(\mathcal{A})$ is a

quasi-isomorphism if the mapping cone

$$\text{cone}(f^\bullet) := \begin{array}{ccccccc} \cdots & \longrightarrow & Y^i \oplus X^{i+1} & \xrightarrow{\begin{pmatrix} d_Y^i & f^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix}} & Y^{i+1} \oplus X^{i+2} & \longrightarrow \cdots \\ & & i & & i+1 & & \end{array}$$

is isomorphic in $H(\mathcal{A})$ to an acyclic complex.

"homotopy equivalent"

Example Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in \mathcal{A} , then

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & 0 & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow g & & \downarrow & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Z & \longrightarrow & 0 & \longrightarrow \cdots \end{array} \quad \text{is a quasi-isomorphism.}$$

In particular

$$\begin{aligned} \text{cone} & \left(\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow \cdots \\ & & & & & & \downarrow f & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow \cdots \end{array} \right) \\ & = \left(\cdots \longrightarrow 0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow \cdots \right) \end{aligned}$$

is quasi-isomorphic to $(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Z \longrightarrow 0 \longrightarrow \cdots)$

("conflations in \mathcal{A} give rise to distinguished triangles in $D(\mathcal{A})$ ")

Def: The derived category of the exact category \mathcal{A} is the localization

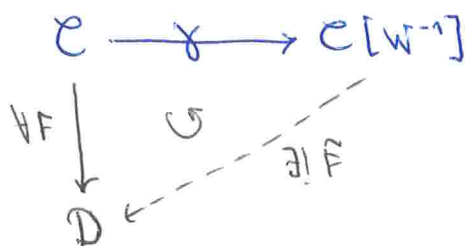
$$\begin{aligned} D(\mathcal{A}) &:= H(\mathcal{A}) [(\text{classes of quasi-isomorphisms})^{-1}] \\ &= C(\mathcal{A}) [(\text{quasi-isomorphisms})^{-1}] \end{aligned}$$

Recall (Universal property of the localization of a category)

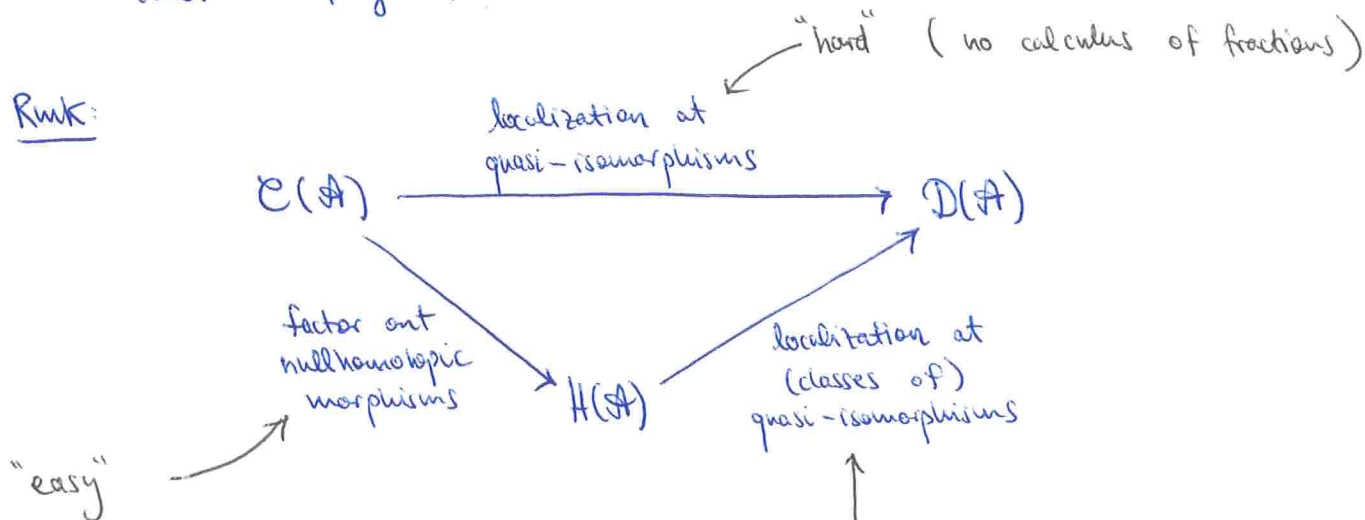
Let \mathcal{C} be a category, $W \subseteq \text{Mor}(\mathcal{C})$ a class of morphisms.

Then $\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$ is a localization of \mathcal{C} at W if

- γ maps W to isomorphisms
- if $F: \mathcal{C} \rightarrow \mathcal{D}$ is any functor which maps W to isomorphisms, then there exists a unique functor \tilde{F} such that $\tilde{F} \circ \gamma = F$.



Remark:



↑
can be made explicit!

I.e. we have a calculus of fractions because we benefit from the structure of a triangulated category on $H(\mathcal{A})$

Reference:

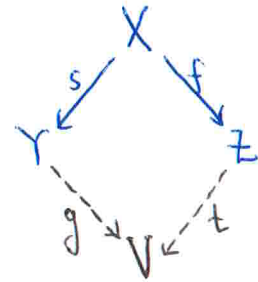
A. Yekutieli: "Derived categories" chapter 6+7

Addendum: Concrete construction of $D(\mathcal{A})$

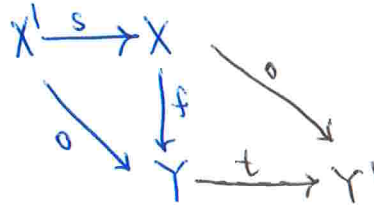
Let $\mathcal{C} := H(\mathcal{A})$ and $\mathcal{W} := \{\text{classes of quasi-isomorphisms}\} \subseteq \text{Mor}(\mathcal{C})$

Lemma (1) \mathcal{W} is closed under composition

(2) $\forall f \in \text{Mor}(\mathcal{C}), s \in \mathcal{W} \exists g \in \text{Mor}(\mathcal{C}), t \in \mathcal{W}$
such that $gs = tf$ (" $f s^{-1} = t^{-1} g$ ")



(3) $\forall s \in \mathcal{W}, f \in \text{Mor}(\mathcal{C})$
such that $fs = 0$
 $\exists t \in \mathcal{W}$ with $tf = 0$



proof One has to use that

$$\mathcal{W} := \left\{ \text{complexes which are homotopy-equivalent to an acyclic complex} \right\} \subseteq \mathcal{C} = H(\mathcal{A})$$

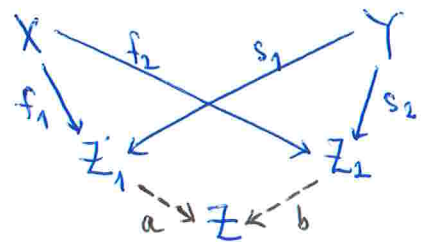
is a triangulated subcategory, e.g. (1) is then a direct consequence of the octahedral axiom. [...]

Construction / Proposition: The following is a localization of \mathcal{C} at \mathcal{W} :

* $\text{Ob}(\mathcal{C}[\mathcal{W}^{-1}]) = \text{Ob}(\mathcal{C})$

* $\text{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(X, Y) := \left\{ (f, s) \mid \begin{array}{ccc} X & & Y \\ & f \searrow & \swarrow s \\ & Z & \end{array} \right\} / \sim$
morphisms in \mathcal{C} and $s \in \mathcal{W}$
 \parallel
 $s^{-1}f = \frac{f}{s}$

with $(f_1, s_1) \sim (f_2, s_2) \iff \exists$ diagram
with $bs_2 \in \mathcal{W}$



* composition via part (2) of the lemma

* $\gamma: \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$
on objects: $X \longmapsto X$
on morphisms: $f \longmapsto (f, \text{id}) = \text{id} \Big/ f$