# ABELIAN MODEL STRUCTURES

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Abstract. This manuscript consists of the author's notes for his talk in "CHARMS Summer School" at Université de Versailles Saint Quentin en Yvelines. The talk was part of a mini-course on "*The Q-shaped derived category*" given by Peter Jørgensen. More specifically, in this talk we introduce the notion of an (abelian) model category and explore the connection between Hovey triples and certain cotorsion pairs in an abelian category $\mathscr{A}$. The main purpose of the talk is to prove that the full subcategory $\mathscr{A}_{cf}$ of cofibrant-fibrant objects is a Frobenius exact category and it's corresponding stable category is the homotopy category $\mathrm{Ho}(\mathscr{A})$ of the model structure on $\mathscr{A}$.

## 1. Motivation

Consider $R$ to be a commutative ring and a multiplicative closed subset $\Sigma$ of $R$. It is known (e.g. [Eis95]) that one can always construct the ring of fractions $R[\Sigma^{-1}]$ in a universal way, given by the canonical map $R \longrightarrow R[\Sigma^{-1}]$. Notice that if $R$ is no longer commutative and $\Sigma$ is an arbitrary subset of $R$ one can still construct the localisation ring but loses control of the elements, as now they form strings $r_1 r_2 r_3 \dots r_k$ with $r_i \in R$ or $r_i \in \Sigma$. If $\Sigma$ has "nice" properties though (i.e. it is an Ore set or a denominator set) one can represent the elements of $R[\Sigma^{-1}]$ as fractions $rs^{-1}$, with $r \in R$ and $s \in \Sigma$.

Notice that a ring $R$ is a pre-additive category with a single object. More generally, consider a category $\mathscr{C}$ and $S \subset \mathrm{Mor}(\mathscr{C})$, where $\mathrm{Mor}(\mathscr{C})$ is the category whose objects are morphisms in $\mathscr{C}$ and the Hom sets, $\mathrm{Hom}_{\mathrm{Mor}(\mathscr{C})}(f, g)$, consist of pairs $(a, b)$ of morphisms in $\mathscr{C}$, such that the following square commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{g} & W \end{array}$$

**Theorem 1.1** ( [GZ12] ). Let $\mathscr{C}$ be a category and $S \subset \mathrm{Mor}(\mathscr{C})$. There is a construction of a category $\mathscr{C}[S^{-1}]$ and a functor $Q\colon \mathscr{C} \longrightarrow \mathscr{C}[S^{-1}]$ that maps morphisms of $S$ to isomorphisms and is universal in the sense that, for every functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ which maps morphisms of $S$ to isomorphisms of $\mathscr{D}$, there exists a unique functor $\bar{F}\colon \mathscr{C}[S^{-1}] \longrightarrow \mathscr{D}$ such that $F = \bar{F}Q$. In symbols, the diagram below commutes.

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{Q} & \mathscr{C}[S^{-1}] \\ {\scriptstyle F}\downarrow & \swarrow{\scriptstyle \exists!\bar{F}} & \\ \mathscr{D} & & \end{array}$$

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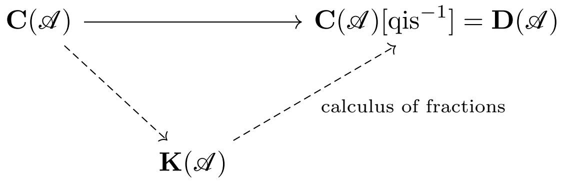
However, in this constructions several problems might occur. In particular one has to consider the following:

- Morphisms in $\mathscr{C}[S^{-1}]$ are hard to control. In particular the form strings of morphisms $f_1 f_2 f_3 \dots f_r$, with $f_i \in \mathrm{Mor}(\mathscr{C})$ or $f_i \in S$.
- More disturbingly, each individual Hom might not even form a set, but a proper class.

Fortunately, the framework of model categories provide a fruitful setup to resolve these problems and a way to better control such categories.

**Example 1.2.** Let $\mathscr{A}$ be an abelian category and $\mathbf{C}(\mathscr{A})$ the category of complexes of $\mathscr{A}$-objects. In Homological Algebra, usually one is interested in "grouping" objects with the same (co-)homology. In particular, we call a morphism $f\colon X \longrightarrow Y$ in $\mathbf{C}(\mathscr{A})$ a **quasi-isomorphism** if the induced morphism in the (co-)homology level $\mathrm{H}^n(f)\colon \mathrm{H}^n(X) \longrightarrow \mathrm{H}^n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$. So one needs to build the category $\mathbf{C}(\mathscr{A})[\mathrm{qis}^{-1}]$, where quasi-isomorphisms are formally inverted. However, as it is known for this purpose it is usually preferred to pass through the intermediate step of the homotopy category $\mathbf{K}(\mathscr{A})$, where we can perform calculus of fractions in the sense of [GZ12]. Schematically:

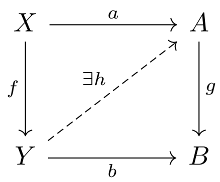



In this manuscript we wish to go directly from $\mathbf{C}(\mathscr{A})$ to the derived category $\mathbf{D}(\mathscr{A})$, using the language of model categories. ✓

## 2. Model Categories

At this point we are ready to introduce the main notion of the manuscript, that is the notion of a model category. Model categories where introduced in [Qui67] and studied extensively in [Hov99]. However we take a slightly different and more modern approach on the subject, based on the survey of Šťovíček [Šťo13, Section 4].

**Definition 2.1.** Let $\mathscr{A}$ be a category and $f\colon X \longrightarrow Y$, $g\colon A \longrightarrow B$ be two morphisms in $\mathscr{A}$. We write $f \,\square\, g$ if for any commutative square




there exists a *lift* $h\colon Y \longrightarrow A$, such that $gh = b$ and $hf = a$. We say that $f$ has the **left lifting property** with respect to $g$ and conversely, that $g$ has the **right lifting property** with respect to $f$.

Let $\mathcal{S}$ be a subset of $\mathrm{Mor}(\mathscr{A})$. We denote the following classes:

$$\mathcal{S}^{\square} = \{g \in \mathrm{Mor}(\mathscr{A})\colon f \,\square\, g, \forall f \in \mathcal{S}\} \quad \text{and} \quad {}^{\square}\mathcal{S} = \{f \in \mathrm{Mor}(\mathscr{A})\colon f \,\square\, g, \forall g \in \mathcal{S}\}$$

It is straight forward to check that $\mathcal{S}^{\square}$ is closed under (existing) pullbacks, while ${}^{\square}\mathcal{S}$ is closed under (existing) pushouts.

**Definition 2.2.** A pair of classes $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathscr{A}$ is called a **weak factorisation system** (WFS) if the following conditions hold.

(1) $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts. That is, every time we have the following commutative diagram

$$\begin{array}{ccccc} & & \overset{1_X}{\dashrightarrow} & & \\ X & \longrightarrow & A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \longrightarrow & B & \longrightarrow & Y \\ & & \underset{1_Y}{\dashrightarrow} & & \end{array}$$

and $g$ is in $\mathcal{L}$ (or $\mathcal{R}$) then $f$ is also in $\mathcal{L}$ (or $\mathcal{R}$).

(2) $\mathcal{L} \,\square\, \mathcal{R}$, that is for every $l \in \mathcal{L}$ and every $r \in \mathcal{R}$, $l \,\square\, r$.

(3) $\mathcal{R} \circ \mathcal{L} = \operatorname{Mor}(\mathscr{A})$, that is, for every morphism $f \in \operatorname{Mor}(\mathscr{A})$, there exist $l \in \mathcal{L}$ and $r \in \mathcal{R}$, such that $f = rl$.

**Remark 2.3.** If the factorisation in the last axiom is functorial, in the sense that there exist functors $R, L \colon \operatorname{Mor}(\mathscr{A}) \longrightarrow \operatorname{Mor}(\mathscr{A})$, such that $f = R(f) \circ L(f)$, we say we have a functorial weak factorisation system.

**Definition 2.4.** Let $\mathscr{A}$ be a category and cof, fib, weq be three subcategories of $\operatorname{Mor}(\mathscr{A})$. We say that (cof, fib, weq) define a **model structure** on $\mathscr{A}$ if the following hold.

(1) weq satisfies the *2-out-of-3* property, that is for each two morphism $f, g$ that the composition $gf$ exists, if two of the morphisms are in weq, so is the third one;

(2) weq is closed under retractions (thus contains all isomorphisms);

(3) $(\mathrm{cof}, \mathrm{fib} \cap \mathrm{weq})$ and $(\mathrm{cof} \cap \mathrm{weq}, \mathrm{fib})$ are weak factorisation systems.

A **model category** is a category $\mathscr{A}$ with finite limits and colimits endowed with a model structure (cof, fib, weq). The morphisms of cof are called **cofibrations**, the morphisms in fib are called **fibrations** and the morphisms in weq are called **weak-equivalences**.

Loosely speaking, one can consider the following motto: "*every time that one has a class of morphisms, that are not isomorphisms but one wishes they were, there must be a model structure in the background with these morphisms as weak-equivalences*". Moreover, for the experts, it might already be apparent that there is a connection between model structures and cotorsion pairs, only by the axioms of the definition.

**Definition 2.5.** Let $\mathscr{A}$ be a model category with initial object $\emptyset$ and terminal object $*$.

(1) An object $X \in \mathscr{A}$ is called **cofibrant** if $\emptyset \longrightarrow X$ is a cofibration.

(2) An object $X \in \mathscr{A}$ is called **fibrant** if $X \longrightarrow *$ is a fibration.

Moreover, for each object $X \in \mathscr{A}$ there exist factorisations:

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ & {\scriptstyle a \in \mathrm{cof}} \searrow \quad \nearrow {\scriptstyle b \in \mathrm{fib} \cap \mathrm{weq}} & \\ & \mathcal{C}(X) & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ & {\scriptstyle \bar{a} \in \mathrm{cof} \cap \mathrm{weq}} \searrow \quad \nearrow {\scriptstyle b \in \mathrm{fib}} & \\ & \mathcal{F}(X) & \end{array}$$

where $\mathcal{C}(X)$ is cofibrant and is called the **cofibrant replacement** of $X$ and $\mathcal{F}(X)$ is fibrant and called the **fibrant replacement** of $X$.

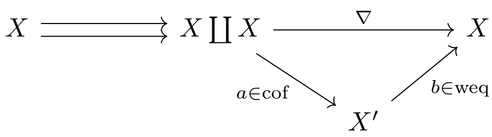
**Remark 2.6.** In the case that (cof, fib $\cap$ weq) and (cof $\cap$ weq, fib) are functorial weak factorisation systems, then the cofibrant and fibrant replacements are functorial as well. That is, $\mathcal{C}(-)$ and $\mathcal{F}(-)$ are functors.

## 3. Homotopy Categories

Consider $\mathscr{A}$ to be a model category. The primary concepts of model categories is to formally invert the weak-equivalences and pass to the localisation category $\mathscr{A}[\text{weq}^{-1}]$, where they are isomorphisms. As we explained in the first section, these categories often are hard to control, however in the model category setup, one has a nice description of them via the Fundamental Theorem of Model Categories of Quillen. For what follows, we denote with $\mathscr{A}_{cf}$ the full subcategory of $\mathscr{A}$ consisting of cofibrant-fibrant objects.

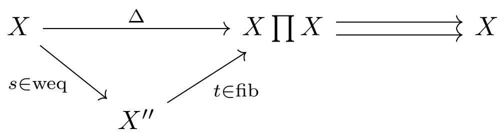
**Definition 3.1.** Let $\mathscr{A}$ be a model category and $X \in \mathscr{A}$.

(1) A **cylinder object** for $X$ is an object $X' \in \mathscr{A}$ and a factorisation




where $\nabla$ is the co-diagonal map.

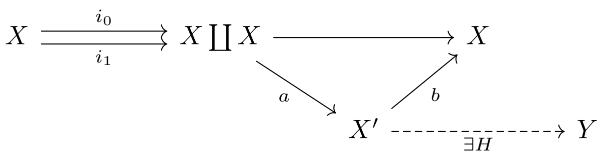
(2) A **path object** for $X$ is an object $X'' \in \mathscr{A}$ and a factorisation




where $\Delta$ is the diagonal map.

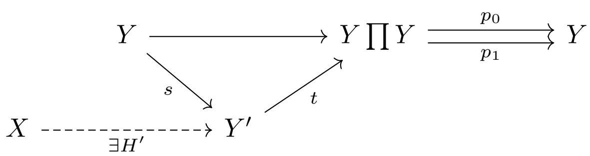
**Definition 3.2.** Let $\mathscr{A}$ be a model category and $f, g \colon X \longrightarrow Y$ be two morphisms in $\mathscr{A}$.

(1) We say that $f$ is **left homotopic** to $g$ and we write $f \overset{l}{\sim} g$ if there exists a cylinder




and a morphisms $H \colon X' \longrightarrow Y$ such that $Hai_0 = f$ and $Hai_1 = g$.

(2) We say that $f$ is **right homotopic** to $g$ and we write $f \overset{r}{\sim} g$ if there exists a path

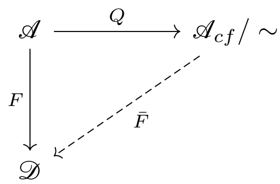



and a morphism $H' \colon X \longrightarrow Y'$ such that $p_0 tH' = f$ and $p_1 tH' = g$.

(3) We say that $f$ is **homotopic** to $g$ if it is both left and right homotopic.

We denote with $\sim$ the homotopy relation on $\mathscr{A}$. One can straightforwardly confirm that the homotopy relation is an equivalence relation on the Hom sets of $\mathscr{A}_{cf}$. In particular, the left and right homotopy coincide in $\mathscr{A}_{cf}$ and the homotopy relation is even compatible with the composition law of $\mathscr{A}$. So we can form the category $\mathscr{A}_{cf}/\sim$ with the same object as $\mathscr{A}_{cf}$ and Homs given by $\mathrm{Hom}_{\mathscr{A}_{cf}/\sim}(X,Y) = \mathrm{Hom}_{\mathscr{A}_{cf}}(X,Y)/\sim$.

**Theorem 3.3** (Fundamental Theorem of Model Categories)**.** Let $\mathscr{A}$ be a model category and $\mathscr{A}_{cf}$ and $\mathscr{A}_{cf}/\sim$ as above. The category $\mathcal{A}_{cf}/\sim$ is equivalent to the **homotopy category** $\mathrm{Ho}(\mathscr{A}) = \mathscr{A}[\mathrm{weq}^{-1}]$ and admits a canonical functor $Q\colon \mathscr{A} \longrightarrow \mathscr{A}_{cf}/\sim$ that maps an object to its cofibrant-fibrant replacement and satisfies the following universal property: for every functor $F\colon \mathscr{A} \longrightarrow \mathscr{D}$ that maps weak-equivalences to isomorphisms, there exists a functor $\bar{F}\colon \mathscr{A}_{cf}/\sim \longrightarrow \mathscr{D}$, such that $F = \bar{F}Q$. Diagrammatically, the following triangle commutes.




## 4. Abelian Model Categories

The theory we developed in the previous sections can be applied to any category. However, in the special case that $\mathscr{A}$ is an abelian (bicomplete) category, one has a better description of the cofibrations and fibrations, in terms of kernels and cokernels.

**Definition 4.1.** An **abelian model category** is a bicomplete[1] abelian category $\mathscr{A}$ endowed with a model structure, such that

(1) A morphism $f$ is a cofibration if and only if, $f$ is a monomorphism and $\operatorname{coker} f$ is a cofibrant object.
(2) A morphism $f$ is a fibration if and only if, $f$ is an epimorphism and $\ker f$ is a fibrant object.

The principal idea of Hovey in [Hov02] is that one can work with objects instead of morphisms which are more controllable. We denote by

$$\mathsf{C} = \{X \in \mathscr{A} \colon 0 \longrightarrow X \in \mathrm{cof}\}$$
$$\mathsf{F} = \{X \in \mathscr{A} \colon X \longrightarrow 0 \in \mathrm{fib}\}$$
$$\mathsf{W} = \{X \in \mathscr{A} \colon 0 \longrightarrow X \in \mathrm{weq}\}$$

the classes of cofibrant, fibrant and trivial objects, respectively. Recall the following definition.

**Definition 4.2.** A pair $(\mathscr{X}, \mathscr{Y})$ in an abelian category $\mathscr{A}$ is called a **cotorsion pair** if $\mathscr{X} = {}^{\perp_1}\mathscr{Y}$ and $\mathscr{Y} = \mathscr{X}^{\perp_1}$. More concretely, if

- $X \in \mathscr{X}$ if and only if $\mathrm{Ext}^1_{\mathscr{A}}(X,Y) = 0$, for all $Y \in \mathscr{Y}$ and
- $Y \in \mathscr{Y}$ if and only if $\mathrm{Ext}^1_{\mathscr{A}}(X,Y) = 0$, for all $X \in \mathscr{X}$.

The cotorsion pair is called **complete** if for every object $A \in \mathscr{A}$, there exist short exact sequences

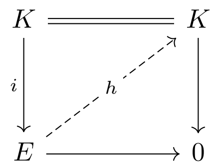
$$0 \longrightarrow Y \longrightarrow X \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow A \longrightarrow Y \longrightarrow X \longrightarrow 0$$

with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. The first short exact sequence is called **special $\mathscr{X}$-pre-cover**, while the second one is called **special $\mathscr{Y}$-pre-envelope**.

[1] A bicomplete abelian category admits all small limits and colimits

**Theorem 4.3** ( [Hov02] )**.** Consider an abelian model category $\mathscr{A}$ and the subclasses $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ defined as above. Then, $(\mathsf{C}, \mathsf{F} \cap \mathsf{W})$ and $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ are complete cotorsion pairs.

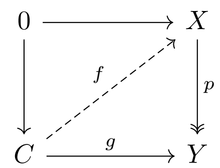
*Sketch of the Proof.* We give the sketch of the proof in steps for the pair $(\mathsf{C}, \mathsf{F} \cap \mathsf{W})$.
**Step 1:** Assume that $C \in \mathsf{C}$ and prove that $\mathrm{Ext}^1_{\mathscr{A}}(C, K) = 0$, for all $K \in \mathsf{F} \cap \mathsf{W}$. Indeed, let $0 \longrightarrow K \xrightarrow{i} E \longrightarrow C \longrightarrow 0$ be an extension of $C$ by $K$. We claim that it splits. Since $C$ is cofibrant $i \colon K \longrightarrow E$ is a cofibration and thus it has the left lifting property with respect to trivial fibrations.




Therefore, it exists a lift $h \colon E \longrightarrow K$ such that $hi = 1_K$, which implies that indeed the extension is trivial.
**Step 2:** Assume now that $\mathrm{Ext}^1_{\mathscr{A}}(C, K) = 0$ for all $K \in \mathsf{F} \cap \mathsf{W}$ and show that $C \in \mathsf{C}$. For this purpose consider the short exact sequence $0 \longrightarrow K \longrightarrow X \xrightarrow{p} Y \longrightarrow 0$, with $K \in \mathsf{F} \cap \mathsf{W}$ and apply the functor $\mathrm{Hom}_{\mathscr{A}}(C, -)$, for an arbitrary $C \in \mathscr{A}$. This yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(C, K) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(C, X) \xrightarrow{p_*} \mathrm{Hom}_{\mathscr{A}}(C, Y) \longrightarrow \mathrm{Ext}^1_{\mathscr{A}}(C, K) = 0$$

which implies that $p_*$ is an epimorphism. Thus, for every $g \colon C \longrightarrow Y$ there is an $f \colon C \longrightarrow X$ such that $p_*(f) = pf = g$. In the language of model categories




the morphism $0 \longrightarrow C$ has the left lifting property with respect to the trivial fibration $p$, hence it is a cofibration and by extension $C$ is cofibrant.
**Step 3:** To show that every object $X \in \mathscr{A}$ admits a special pre-cover, consider the morphism $0 \longrightarrow X$, which by the factorisation axiom factors into a cofibration followed by a trivial fibration, that is there exists the desired short exact sequence

$$0 \longrightarrow K \longrightarrow C \longrightarrow X \longrightarrow 0$$

with $K \in \mathsf{F} \cap \mathsf{W}$ and $C \in \mathsf{C}$. The remaining assertions are left as an exercise for the reader. ∎

**Definition 4.4.** A subcategory $\mathscr{B}$ of $\mathscr{A}$ is called **thick**, if it is closed under direct summands and satisfies the 2-out-of-3 property, that is for each short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathscr{A}$ with two entries in $\mathscr{B}$, the third one is also in $\mathscr{B}$.

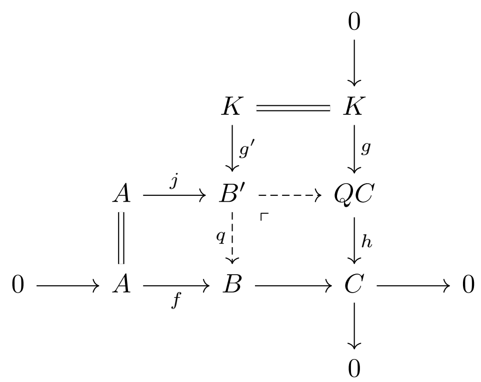
It is straightforward to prove that the class of trivial objects $\mathsf{W}$ is a thick subcategory of the abelian model category $\mathscr{A}$.

**Theorem 4.5** ( [Hov02] )**.** Let $(\mathsf{C}, \mathsf{F} \cap \mathsf{W})$ and $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ be complete cotorsion pairs in $\mathscr{A}$ and $\mathsf{W}$ is thick. Then, $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ is an (abelian) **Hovey triple** on $\mathscr{A}$, that is, $\mathscr{A}$ is an abelian model category with $\mathsf{C}, \mathsf{F}$ and $\mathsf{W}$ as cofibrant, fibrant and trivial objects, respectively.

*Idea of the Proof.* The proof of this Theorem is extensive and takes up a whole section in [Hov02]. We prove only the factorisation axiom in which it is apparent that the completeness of the cotorsion pairs is necessary and we note that the thickness of $\mathsf{W}$ is mandatory in order to prove the 2-out-of-3 property of weq.

We prove that every $f \in \mathrm{Mor}(\mathscr{A})$ factors as $f = qj = pi$, where $j \in \mathrm{cof}$, $i \in \mathrm{fib}$, $q \in \mathrm{fib} \cap \mathrm{weq}$ and $p \in \mathrm{weq} \cap \mathrm{cof}$. In particular we only prove that $f$ factors as $f = qj$. To do so, we consider the following cases.

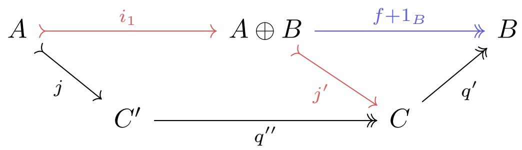
**Monomorphism:** Consider that $f$ is a monomorphism with $C = \operatorname{coker} f$. By the completeness of the cotorsion pair $(\mathsf{C}, \mathsf{F} \cap \mathsf{W})$ there is an object $QC \in \mathsf{W}$ and an epimorphism $QC \longrightarrow C$ with trivial fibrant kernel $K \in \mathsf{F} \cap \mathsf{W}$, that is, we have a short exact sequence $0 \longrightarrow K \longrightarrow QC \longrightarrow C \longrightarrow 0$. So, we end up with a diagram




that "begs to become" a pullback diagram. So $f = qj$, where $q$ is a trivial fibration since $h$ is a trivial fibration and $j$ is a cofibration since it's a monomorphism with cofibrant cokernel $QC$.

**Epimorphism:** Follows by the dual arguments.

Now assume that $f$ is an arbitrary morphism. Then we can re-write it as




First we factor the epimorphism $f+1_B$ as $f+1_B = q'j'$ and then the monomorphism $j'i_1$ as $j'i_1 = q''j$. It follows that $f = qj$, where $q = q'q''$. ∎

We conclude that the abelian model structures on an abelian category $\mathscr{A}$ are in one-to-one correspondence with complete cotorsion pairs $(\mathsf{C}, \mathsf{F} \cap \mathsf{W})$ and $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ with $\mathsf{W}$ thick in $\mathscr{A}$. In the case of abelian model categories, the associated homotopy category, behaves nicely.

**Theorem 4.6** ( [Gil16] )**.** Let $\mathscr{A}$ be a bicomplete abelian category and $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ a Hovey triple on $\mathscr{A}$. The following hold.

(1) The full subcategory $\mathscr{A}_{cf} = \mathsf{C} \cap \mathsf{F}$ is a Frobenius exact category, with the inherent exact structure from $\mathscr{A}$;
(2) The class of projective/injective objects on $\mathscr{A}_{cf}$ is the core $\omega = \mathsf{C} \cap \mathsf{W} \cap \mathsf{F}$;
(3) The inclusion $\mathscr{A}_{cf} \longrightarrow \mathscr{A}$ induces an equivalence of categories

$$\frac{\mathscr{A}_{cf}}{\mathsf{C} \cap \mathsf{W} \cap \mathsf{F}} \xrightarrow{\sim} \mathrm{Ho}(\mathscr{A}) = \mathscr{A}[\mathrm{weq}^{-1}].$$

In particular, $\mathrm{Ho}(\mathscr{A})$ is triangulated.

*Sketch of the Proof.* The full proof can be found in [Gil16] and in greater generality in [Gil11]. We provide the main idea.

Notice that since $\mathscr{A}_{cf}$ is closed under extensions, it inherits an exact structure from $\mathscr{A}$. In particular this exact structure is Frobenius. Assume the short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathscr{A}_{cf}$. If either of $X$ or $Z$ is in $\omega$, the short exact sequence splits. Thus, every object in $\omega$ is projective and injective with respect to the exact structure. Assume now that $I \in \mathscr{A}_{cf}$ is injective with respect to the exact structure on $\mathscr{A}_{cf}$. By the completeness of the cotorsion pair $\mathsf{C}, \mathsf{F} \cap \mathsf{W})$, there exists a (split) short exact sequence $0 \longrightarrow I \longrightarrow W \longrightarrow C \longrightarrow 0$, with $W \in \mathsf{W} \cap \mathsf{F}$ and $C \in \mathsf{C}$. Since the model structure is hereditary, one has that $C \in \mathsf{C} \cap \mathsf{F}$ and moreover $W \in \omega$. So, $I \in \omega$ as $\omega$ is closed under direct summands and the sequence above splits. Furthermore, replacing $I$ with an arbitrary $X \in \mathscr{A}_{cf}$, the same argument yields that $\mathcal{A}_{cf}$ has enough injectives. Dually one can prove the same for the projectives. The third assertion of the Theorem follows by the Fundamental Theorem of Model Categories 3.3. Finally, $\mathrm{Ho}(\mathscr{A})$ is triangulated by Happel's Theorem (*c.f.* [Hap88]). ■

## 5. Examples

We conclude our manuscript with a familiar example, decorated in the framework of model categories.

**Example 5.1.** Let $R$ be an associative unital ring and $\mathscr{A} = \mathbf{C}(R)$ the category of complexes of $R$-modules. There are two model structures on $\mathscr{A}$.

**Injective model structure:** In the injective model structure one has the classes $\mathsf{C} = \mathscr{A}$, $\mathsf{W} = \mathrm{Ac}(\mathscr{A})$ of acyclic complexes of $R$-modules and $\mathsf{F} = \text{DG-Inj}(\mathscr{A})$ of semi-injective complexes, as the classes of cofibrant, trivial and fibrant objects, respectively. Recall that a complex $I \in \mathbf{C}(R)$ is called homotopically injective if it's terms are injective $R$-modules and for every acyclic complex $N \in \mathbf{C}(R)$, $\mathrm{Hom}_{\mathbf{K}(R)}(N, I) = 0$. The cotorsion pairs in our case are

$$(\mathsf{C}, \mathsf{F} \cap \mathsf{W}) = (\mathscr{A}, \mathrm{Inj}(\mathscr{A})) \quad \text{and} \quad (\mathsf{C} \cap \mathsf{W}, \mathsf{F}) = (\mathrm{Ac}(\mathscr{A}), \text{DG-Inj}(\mathscr{A}))$$

It is routine to check that they are complete and that $\mathrm{Ac}(\mathscr{A})$ is thick in $\mathscr{A}$. Thus by Theorem 4.5 we have an abelian model structure on $\mathscr{A}$, whose weak-equivalences are the quasi-isomorphisms. The associated homotopy category is

$$\mathrm{Ho}(\mathscr{A}) = \mathbf{C}(R)[\mathrm{qis}^{-1}] \cong \mathbf{D}(R)$$

the derived category of the ring $R$ (as expected by Krause's recollement situation in [Kra10]).

**Projective model structure:** Dually one can define the projective model structure on $\mathscr{A} = \mathbf{C}(R)$ by taking $\mathsf{C} = \text{DG-Prj}(\mathscr{A})$, $\mathsf{W} = \mathrm{Ac}(\mathscr{A})$ and $\mathsf{F} = \mathscr{A}$. This model structure has the same weak-equivalences as the injective model structure and by extension the same homotopy category, that is, the derived category $\mathbf{D}(R)$. ✓

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